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In Junior Certificate Mathematics, learners build on their primary school experience and continue to develop their understanding of data analysis by collecting, representing, describing, and interpreting numerical data. By carrying out a complete investigation, from formulating a question through to drawing conclusions from data, learners gain an understanding of data analysis as a tool for learning about the world. Work in this strand focuses on engaging learners in this process of data investigation: posing questions, collecting data, analysing and interpreting this data in order to answer questions.

Learners advance in their knowledge of chance from primary school to deal more formally with probability. The *Common Introductory Course* (see appendix), which draws on a selection of learning outcomes from this strand, enables learners to begin the process of engaging in a more formal manner with the concepts and processes involved.

**Topic descriptions and learning outcomes listed in bold text are for Higher Level only.**

In the course of studying this strand the learner will

- use a variety of methods to represent their data
- explore concepts that relate to ways of describing data
- develop a variety of strategies for comparing data sets
- complete a data investigation of their own
- encounter the language and concepts of probability.



## Strand 1: Statistics and Probability

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
<b>1.1 Counting</b>	Listing outcomes of experiments in a systematic way.	<ul style="list-style-type: none"> <li>– list all possible outcomes of an experiment</li> <li>– apply the fundamental principle of counting</li> </ul>
<b>1.2 Concepts of probability</b>	<p>The probability of an event occurring: student progress from informal to formal descriptions of probability.</p> <p>Predicting and determining probabilities. Difference between experimental and theoretical probability.</p>	<ul style="list-style-type: none"> <li>– decide whether an everyday event is likely or unlikely to occur</li> <li>– recognise that probability is a measure on a scale of 0-1 of how likely an event is to occur</li> <li>– <b>use set theory to discuss experiments, outcomes, sample spaces</b></li> <li>– use the language of probability to discuss events, including those with equally likely outcomes</li> <li>– estimate probabilities from experimental data</li> <li>– recognise that, if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability</li> <li>– associate the probability of an event with its long-run, relative frequency</li> </ul>
<b>1.3 Outcomes of simple random processes</b>	Finding the probability of equally likely outcomes.	<ul style="list-style-type: none"> <li>– apply the principle that, in the case of equally likely outcomes, the probability is given by the number of outcomes of interest divided by the total number of outcomes (examples using coins, dice, spinners, urns with different coloured objects, playing cards, etc.)</li> <li>– <b>use binary / counting methods to solve problems involving successive random events where only two possible outcomes apply to each event</b></li> </ul>
<b>1.4 Statistical reasoning with an aim to becoming a statistically aware consumer</b>	The use of statistics to gather information from a selection of the population with the intention of making generalisations about the whole population. They consider situations where statistics are misused and learn to evaluate the reliability and quality of data and data sources.	<ul style="list-style-type: none"> <li>– engage in discussions about the purpose of statistics and recognise misconceptions and misuses of statistics</li> <li>– work with different types of data (categorical/numerical/ordinal discrete/continuous) in order to clarify the problem at hand</li> <li>– <b>evaluate reliability of data and data sources</b></li> </ul>
<b>1.5 Finding, collecting and organising data</b>	Formulating a statistics question based on data that vary allows for distinction between different types of data.	<ul style="list-style-type: none"> <li>– clarify the problem at hand</li> <li>– formulate one (or more) questions that can be answered with data</li> <li>– explore different ways of collecting data</li> <li>– generate data, or source data from other sources including the internet</li> <li>– <b>select a sample (Simple Random Sample)</b></li> <li>– <b>recognise the importance of representativeness so as to avoid biased samples</b></li> <li>– design a plan and collect data on the basis of above knowledge</li> <li>– summarise data in diagrammatic form including spread sheets</li> </ul>

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
<b>1.6</b> <b>Representing data graphically and numerically</b>	<p>Methods of representing data.</p> <p>Students develop a sense that data can convey information and that organising data in different ways can help clarify what the data have to tell us. They see a data set as a whole and so are able to use fractions, quartiles and median to describe the data.</p> <p><b>Mean of a grouped frequency distribution.</b></p>	<p>Graphical</p> <ul style="list-style-type: none"> <li>– select appropriate graphical or numerical methods to describe the sample (univariate data only)</li> <li>– evaluate the effectiveness of different displays in representing the findings of a statistical investigation conducted by others</li> <li>– use pie charts, bar charts, line plots, histograms (equal intervals), stem and leaf plots to display data</li> <li>– <b>use back-to-back stem and leaf plots to compare data sets</b></li> </ul> <p>Numerical</p> <ul style="list-style-type: none"> <li>– use a variety of summary statistics to analyse the data: central tendency – mean, median, mode</li> <li>– variability – range</li> <li>– <b>use stem plots to calculate quartiles and inter-quartile range</b></li> </ul>
<b>1.7</b> <b>Analysing, interpreting and drawing conclusions from data</b>	<p>Drawing conclusions from data; limitations of conclusions.</p>	<ul style="list-style-type: none"> <li>– interpret graphical summaries of data</li> <li>– relate the interpretation to the original question</li> <li>– <b>recognise how sampling variability influences the use of sample information to make statements about the population</b></li> <li>– <b>draw conclusions from graphical and numerical summaries of data, recognising assumptions and limitations</b></li> </ul>
<b>Students learn about</b>	<b>Students should be able to</b>	
<b>1.8</b> <b>Synthesis and problem-solving skills</b>	<ul style="list-style-type: none"> <li>– explore patterns and formulate conjectures</li> <li>– explain findings</li> <li>– justify conclusions</li> <li>– communicate mathematics verbally and in written form</li> <li>– apply their knowledge and skills to solve problems in familiar and unfamiliar contexts</li> <li>– analyse information presented verbally and translate it into mathematical form</li> <li>– devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.</li> </ul>	

## Strand 2: Geometry and Trigonometry

The synthetic geometry covered in Junior Certificate Mathematics is selected from the *Geometry Course for Post-primary School Mathematics*, including terms, definitions, axioms, propositions, theorems, converses and corollaries. The formal underpinning for the system of post-primary geometry is that described by Barry (2001)<sup>1</sup>.

The geometrical results listed in the following pages should first be encountered by learners through investigation and discovery. The Common Introductory Course will enable learners to link formal geometrical results to their study of space and shape in primary mathematics. Learners are asked to accept these results as true for the purpose of applying them to various contextualised and abstract problems. They should come to appreciate that certain features of shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features or results can be established in a formal manner through logical proof. Even at the investigative stage, ideas involved in mathematical proof can be developed. Learners should become familiar with the formal proofs of the specified theorems (some of which are examinable at Higher level).

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<sup>1</sup> P.D. Barry. *Geometry with Trigonometry*, Horwood, Chichester (2001)

It is envisaged that learners will engage with dynamic geometry software, paper folding and other active investigative methods.

**Topic descriptions and learning outcomes listed in bold text are for Higher Level only.**

In the course of studying this strand the learner will

- recall basic facts related to geometry and trigonometry
- construct a variety of geometric shapes and establish their specific properties or characteristics
- solve geometrical problems and in some cases present logical proofs
- interpret information presented in graphical and pictorial form
- analyse and process information presented in unfamiliar contexts
- select appropriate formulae and techniques to solve problems.

## Strand 2: Geometry and Trigonometry

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
<b>2.1 Synthetic geometry</b>	<p>Concepts (see <i>Geometry Course</i> section 9.1 for OL and <b>10.1 for HL</b>)</p> <p>Axioms (see <i>Geometry Course</i> section 9.3 for OL and <b>10.3 for HL</b>):</p> <ol style="list-style-type: none"> <li>[Two points axiom] There is exactly one line through any two given points.</li> <li>[Ruler axiom] The properties of the distance between points</li> <li>[Protractor Axiom] The properties of the degree measure of an angle</li> <li>Congruent triangles (SAS, ASA and SSS)</li> <li>[Axiom of Parallels] Given any line <math>l</math> and a point <math>P</math>, there is exactly one line through <math>P</math> that is parallel to <math>l</math>.</li> </ol> <p>Theorems: [Formal proofs are not examinable at OL]</p> <ol style="list-style-type: none"> <li>Vertically opposite angles are equal in measure.</li> <li>In an isosceles triangle the angles opposite the equal sides are equal. Conversely, if two angles are equal, then the triangle is isosceles.</li> <li>If a transversal makes equal alternate angles on two lines then the lines are parallel, (and converse).</li> <li>The angles in any triangle add to <math>180^\circ</math>.</li> <li>Two lines are parallel if and only if, for any transversal, the corresponding angles are equal.</li> <li>Each exterior angle of a triangle is equal to the sum of the interior opposite angles.</li> <li>In a parallelogram, opposite sides are equal and opposite angles are equal (and converses).</li> <li>The diagonals of a parallelogram bisect each other.</li> <li><b>11. If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.</b></li> <li><b>12. Let ABC be a triangle. If a line <math>l</math> is parallel to BC and cuts [AB] in the ratio <math>s:t</math>, then it also cuts [AC] in the same ratio (and converse).</b></li> <li><b>13. If two triangles are similar, then their sides are proportional, in order (and converse) [statements only at OL].</b></li> <li>[Theorem of Pythagoras] In a right-angled triangle the square of the hypotenuse is the sum of the squares of the other two sides.</li> <li>If the square of one side of a triangle is the sum of the squares of the other two sides, then the angle opposite the first side is a right angle.</li> <li><b>19. The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.</b></li> </ol> <p>[Formal proofs of theorems 4, 6, 9, 14 and 19 are examinable at Higher level.]</p>	<ul style="list-style-type: none"> <li>– recall the axioms and use them in the solution of problems</li> <li>– <b>use the terms: theorem, proof, axiom, corollary, converse and implies</b></li> <li>– apply the results of all theorems, converses and corollaries to solve problems</li> <li>– <b>prove the specified theorems</b></li> </ul>

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
	<p>Corollaries:</p> <ol style="list-style-type: none"> <li><b>1. A diagonal divides a parallelogram into 2 congruent triangles.</b></li> <li><b>2. All angles at points of a circle, standing on the same arc, are equal, (and converse).</b></li> <li>Each angle in a semi-circle is a right angle.</li> <li>If the angle standing on a chord [BC] at some point of the circle is a right-angle, then [BC] is a diameter.</li> <li><b>5. If ABCD is a cyclic quadrilateral, then opposite angles sum to <math>180^\circ</math>, (and converse).</b></li> </ol> <p>Constructions:</p> <ol style="list-style-type: none"> <li>Bisector of a given angle, using only compass and straight edge.</li> <li>Perpendicular bisector of a segment, using only compass and straight edge.</li> <li><b>3. Line perpendicular to a given line l, passing through a given point not on l.</b></li> <li>Line perpendicular to a given line l, passing through a given point on l.</li> <li>Line parallel to a given line, through a given point.</li> <li>Division of a line segment into 2 or 3 equal segments, without measuring it.</li> <li><b>7. Division of a line segment into any number of equal segments, without measuring it.</b></li> <li>Line segment of a given length on a given ray.</li> <li>Angle of a given number of degrees with a given ray as one arm.</li> <li>Triangle, given lengths of three sides</li> <li>Triangle, given SAS data</li> <li>Triangle, given ASA data</li> <li>Right-angled triangle, given the length of the hypotenuse and one other side.</li> <li>Right-angled triangle, given one side and one of the acute angles (several cases).</li> <li>Rectangle, given side lengths.</li> </ol>	<p>– complete the constructions specified</p>
<p><b>2.2 Transformation Geometry</b></p>	<p>Translations, central symmetry and axial symmetry.</p>	<p>– locate axes of symmetry in simple shapes</p> <p>– recognise images of points and objects under translation, central symmetry and axial symmetry (intuitive approach)</p>

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
<b>2.3 Co-ordinate Geometry</b>	<p>Co-ordinating the plane.</p> <p>Properties of lines and line segments including midpoint, slope, distance and the equation of a line in the form.</p> $y - y_1 = m(x - x_1)$ $y = mx + c$ <p><b><math>ax + by + c = 0</math></b> where <math>a, b, c</math>, are integers and <math>m</math> is the slope of the line</p> <p>Intersection of lines.</p> <p><b>Parallel and perpendicular lines and the relationships between the slopes.</b></p>	<ul style="list-style-type: none"> <li>– explore the properties of points, lines and line segments including the equation of a line</li> <li>– find the point of intersection of two lines, <b>including algebraically</b></li> <li>– <b>find the slopes of parallel and perpendicular lines</b></li> </ul>
<b>2.4 Trigonometry</b>	<p>Right-angled triangles: theorem of Pythagoras.</p> <p>Trigonometric ratios</p> <p><b>Trigonometric ratios in surd form for angles of <math>30^\circ</math>, <math>45^\circ</math> and <math>60^\circ</math></b></p> <p><b>Right-angled triangles</b></p> <p><b>Decimal and DMS values of angles.</b></p>	<ul style="list-style-type: none"> <li>– apply the result of the theorem of Pythagoras to solve right-angled triangle problems of a simple nature involving heights and distances</li> <li>– use trigonometric ratios to solve problems involving angles (integer values) between <math>0^\circ</math> and <math>90^\circ</math></li> <li>– <b>solve problems involving surds</b></li> <li>– <b>solve problems involving right-angled triangles</b></li> <li>– <b>manipulate measure of angles in both decimal and DMS forms</b></li> </ul>
<b>Students learn about</b>	<b>Students should be able to</b>	
<b>2.5 Synthesis and problem-solving skills</b>	<ul style="list-style-type: none"> <li>– explore patterns and formulate conjectures</li> <li>– explain findings</li> <li>– justify conclusions</li> <li>– communicate mathematics verbally and in written form</li> <li>– apply their knowledge and skills to solve problems in familiar and unfamiliar contexts</li> <li>– analyse information presented verbally and translate it into mathematical form</li> <li>– devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.</li> </ul>	



# ASSESSMENT

# Assessment

## General principles

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Assessment in education involves gathering, interpreting and using information about the processes and outcomes of learning. It takes different forms and can be used in a variety of ways, such as to test and certify achievement, to determine the appropriate route for learners to take through a differentiated curriculum, or to identify specific areas of difficulty (or strength) for a given learner. While different techniques may be employed for formative, diagnostic and certification purposes, assessment of any kind can improve learning by exerting a positive influence on the curriculum at all levels. To do this it must reflect the full range of curriculum goals.

Assessment should be used as a continuous part of the teaching-learning process and involve learners, wherever possible, as well as teachers in identifying next steps. In this context, the most valuable assessment takes place at the site of learning. Assessment also provides an effective basis for communication with parents in a way that helps them to support their children's learning. Assessment must be valid, reliable and equitable. These aspects of assessment are particularly relevant for national assessment for certification purposes.

## Assessment for certification

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Junior Certificate Mathematics is assessed at Foundation, Ordinary and Higher levels. At Foundation level there is one examination paper. There are two assessment components at Ordinary and Higher level

- Mathematics Paper 1
- Mathematics Paper 2

Differentiation at the point of assessment is achieved through the language level in the examination questions, the stimulus material presented, and the amount of structured support given in the questions, especially for candidates at Foundation level.

The understanding of mathematics that the learner has from Strands 1 and 2 will be assessed through a focus on *concepts and skills* and *context and applications*. Learners will be asked to engage with real life problems and to explain and justify conclusions. In this regard some assessment items will differ from those traditionally presented in examination papers.

In Strands 1 and 2, learners at Foundation level can expect to engage with a variety of tasks, including word problems, but in language that is appropriate to this level. There will be structured support within tasks to assist in progression through a problem. Learners will be expected to give an opinion and to justify and explain their reasoning in some answers. The assessment of Strands 1 and 2 will reflect the changed methodology and active nature of teaching and learning in the classroom.

In Strands 1 and 2, the tasks for learners at Ordinary level will be more challenging than Foundation level tasks and candidates may not receive the same level of structured support in a problem. They will be expected to deal with problem solving in real world contexts and to draw conclusions from answers. The quality of the answering expected will be higher than that at Foundation level.

In Strands 1 and 2, learners at Higher level will be expected to deal with more complex and challenging problems than those at Ordinary level. They will be asked to demonstrate a deeper understanding of concepts and an ability to employ a variety of strategies to solve problems as well as to apply mathematical knowledge. Learners at this level can expect to be tested on Ordinary level learning outcomes but their tasks will be, to an appropriate degree, are more complex and difficult.



# Appendix: Common Introductory Course

The Common Introductory Course is the minimum course to be covered by all learners at the start of junior cycle. The learning outcomes are reflective of those listed in the individual syllabus strands.

Once the introductory course has been completed, teachers can decide which topics to extend or explore to a greater depth, depending on the progress being made by the class group. The order in which topics are taught is left to the discretion of the teacher.

To facilitate a smooth transition for learners from their mathematics education in the primary school, teaching and learning plans for the Common Introductory Course should be developed with reference to the main features of the Primary School Curriculum in Mathematics referred to in the introductory sections of this syllabus.

Strand	Learning outcomes Students should be able to
<b>Strand 1:</b> 1.1 Counting	<ul style="list-style-type: none"> <li>– list outcomes of an experiment</li> <li>– apply the fundamental principle of counting</li> </ul>
<b>Strand 1:</b> 1.2 Concepts of probability It is expected that the conduct of experiments (including simulations), both individually and in groups, will form the primary vehicle through which the knowledge, understanding and skills in probability are developed.	<ul style="list-style-type: none"> <li>– decide whether an everyday event is likely or unlikely to happen</li> <li>– appreciate that probability is a quantity that gives a measure on a scale of 0 - 1 of how likely an event is to occur</li> </ul>
<b>Strand 1:</b> 1.5 Finding, collecting and organising data	<ul style="list-style-type: none"> <li>– pose a question and reflect on the question in the light of data collected</li> <li>– plan an investigation involving statistics</li> <li>– select a sample and appreciate the importance of representativeness so as to avoid biased samples</li> <li>– design a plan and collect data on the basis of above knowledge</li> </ul>
<b>Strand 1:</b> 1.6 Representing data graphically and numerically	<ul style="list-style-type: none"> <li>– select appropriate graphical or numerical methods to describe the sample (univariate data only)</li> <li>– use stem and leaf plots, line plots and bar charts to display data</li> </ul>
<b>Strand 2:</b> 2.1 Synthetic Geometry (see <i>Geometry Course for Post-primary School Mathematics</i> )  The geometrical results should be first encountered through discovery and investigation.	<ul style="list-style-type: none"> <li>– convince themselves through investigation that theorems 1-6 are true</li> <li>– construct               <ol style="list-style-type: none"> <li>1. the bisector of a given angle, using only compass and straight edge</li> <li>2. the perpendicular bisector of a segment, using only compass and straight edge</li> <li>4. a line perpendicular to a given line <math>l</math>, passing through a given point on <math>l</math></li> <li>5. a line parallel to a given line <math>l</math>, through a given point</li> <li>6. divide a line segment into 2, 3 equal segments, without measuring it</li> <li>8. a line segment of given length on a given ray</li> </ol> </li> </ul>
<b>Strand 2:</b> 2.2 Transformation geometry	<ul style="list-style-type: none"> <li>– use drawings to show central symmetry and axial symmetry</li> </ul>
<b>Strand 2:</b> 2.3 Co-ordinate geometry	<ul style="list-style-type: none"> <li>– coordinate the plane</li> <li>– locate points on the plane using coordinates</li> </ul>

Topic	Selected topics from the previous Junior Certificate Mathematics syllabus
<b>Sets</b>	<p>1. Listing of elements of a set. Membership of a set defined by a rule. Universe, subsets. Null set (empty set). Equality of sets.</p> <p>2. Venn Diagrams</p>
<b>Number Systems</b>	<p>1. The set <math>N</math> of natural numbers. Order (<math>&lt;</math>, <math>\leq</math>, <math>&gt;</math>, <math>\geq</math>). Place value. Sets of divisors. Pairs of factors. Prime numbers. Sets of multiples. Lowest common multiple. Highest common factor. Cardinal number of a set. The operations of addition, subtraction, multiplication and division in <math>N</math>. Meaning of <math>a^n</math> for <math>a, n \in N, n \neq 0</math>. Estimation leading to approximate answers.</p>
	<p>2. The set <math>Z</math> of integers. Order (<math>&lt;</math>, <math>\leq</math>, <math>&gt;</math>, <math>\geq</math>). The operations of addition, subtraction, multiplication and division in <math>Z</math>. Use of the number line to illustrate addition, subtraction and multiplication. Meaning of <math>a^n</math> for <math>a \in Z, n \in N, n \neq 0</math>. Estimation leading to approximate answers.</p>
	<p>3. The set <math>Q</math> of rational numbers. Decimals, fractions, percentages. Decimals and fractions plotted on the number line. Rational numbers expressed as decimals. Terminating decimals expressed as fractions. The operations of addition, subtraction, multiplication and division in <math>Q</math>. Rounding off. Significant figures for integer values only. Estimation leading to approximate answers. Ratio and proportion.</p>
	<p>5. The set <math>R</math> of real numbers: Every point on the number line represents a real number. Order (<math>&lt;</math>, <math>\leq</math>, <math>&gt;</math>, <math>\geq</math>).</p>
	<p>6. Commutative and associative properties for addition and multiplication; failure of commutativity and associativity for subtraction and division; distributive property of multiplication over addition. Priority of operations.</p>
<b>Applied arithmetic and measure</b>	<p>1. Perimeter. Area: square, rectangle, triangle.</p>
<b>Algebra</b>	<p>1. Meaning of the terms variable, constant, term, expression, coefficient. Evaluation of expressions.</p> <p>2. Addition and subtraction of simple algebraic expressions of forms such as: <math>(ax + by + c) \pm (dx + ey + f)</math> <math>(ax^2 + bx + c) \pm (dx^2 + ex + f)</math> where <math>a, b, c, d, e, f \in Z</math>.</p> <p>3. Formation and interpretation of number sentences leading to the solution of first degree equations in one variable.</p>

## **Section B**

### **Geometry Course for Post-primary School Mathematics**

This section sets out the agreed course in geometry for both Junior Certificate Mathematics and Leaving Certificate Mathematics. Strand 2 of the relevant syllabus document specifies the learning outcomes at the different syllabus levels.



# Geometry Course for Post-primary School Mathematics

September 2010

## 1 Introduction

The Junior Certificate and Leaving Certificate mathematics course committees of the National Council for Curriculum and Assessment (NCCA) accepted the recommendation contained in the paper [4] to base the logical structure of post-primary school geometry on the level 1 account in Professor Barry's book [1].

To quote from [4]: We distinguish three levels:

Level 1: The fully-rigorous level, likely to be intelligible only to professional mathematicians and advanced third- and fourth-level students.

Level 2: The semiformal level, suitable for digestion by many students from (roughly) the age of 14 and upwards.

Level 3: The informal level, suitable for younger children.

This appendix sets out the agreed course in geometry for post-primary schools. It was prepared by a working group of the NCCA course committees for mathematics and, following minor amendments, was adopted by both committees for inclusion in the syllabus documents. Readers should refer to Strand 2 of the syllabus documents for Junior Certificate and Leaving Certificate mathematics for the range and depth of material to be studied at the different levels. A summary of these is given in sections 9–13 of this appendix.

The preparation and presentation of this appendix was undertaken principally by Anthony O'Farrell, with assistance from Ian Short. Helpful criticism from Stefan Bechluft-Sachs, Ann O'Shea and Richard Watson is also acknowledged.

## 2 The system of geometry used for the purposes of formal proofs

In the following, Geometry refers to plane geometry.

There are many formal presentations of geometry in existence, each with its own set of axioms and primitive concepts. What constitutes a valid proof in the context of one system might therefore not be valid in the context of another. Given that students will be expected to present formal proofs in the examinations, it is therefore necessary to specify the system of geometry that is to form the context for such proofs.

The formal underpinning for the system of geometry on the Junior and Leaving Certificate courses is that described by Prof. Patrick D. Barry in [1]. A properly formal presentation of such a system has the serious disadvantage that it is not readily accessible to students at this level. Accordingly, what is presented below is a necessarily simplified version that treats many concepts far more loosely than a truly formal presentation would demand. Any readers who wish to rectify this deficiency are referred to [1] for a proper scholarly treatment of the material.

Barry's system has the primitive undefined terms **plane**, **point**, **line**,  $<_l$  (**precedes on a line**), **(open) half-plane**, **distance**, and **degree-measure**, and seven axioms:  $A_1$ : about incidence,  $A_2$ : about order on lines,  $A_3$ : about how lines separate the plane,  $A_4$ : about distance,  $A_5$ : about degree measure,  $A_6$ : about congruence of triangles,  $A_7$ : about parallels.

## 3 Guiding Principles

In constructing a level 2 account, we respect the principles about the relationship between the levels laid down in [4, Section 2].

The choice of material to study should be guided by applications (inside and outside Mathematics proper).

The most important reason to study synthetic geometry is to prepare the ground logically for the development of trigonometry, coordinate geometry, and vectors, which in turn have myriad applications.

We aim to keep the account as simple as possible.

We also take it as desirable that the official Irish syllabus should avoid imposing terminology that is nonstandard in international practice, or is used in a nonstandard way.

No proof should be allowed at level 2 that cannot be expanded to a complete rigorous proof at level 1, or that uses axioms or theorems that come later in the logical sequence. We aim to supply adequate proofs for all the theorems, but do not propose that only those proofs will be acceptable. It should be open to teachers and students to think about other ways to prove the results, provided they are correct and fit within the logical framework. Indeed, such activity is to be encouraged. Naturally, teachers and students will need some assurance that such variant proofs will be acceptable if presented in examination. We suggest that the discoverer of a new proof should discuss it with students and colleagues, and (if in any doubt) should refer it to the National Council for Curriculum and Assessment and/or the State Examinations Commission.

It may be helpful to note the following non-exhaustive list of salient differences between Barry’s treatment and our less formal presentation.

- Whereas we may use set notation and we expect students to understand the conceptualisation of geometry in terms of sets, we more often use the language that is common when discussing geometry informally, such as “the point is/lies on the line”, “the line passes through the point”, etc.
- We accept and use a much lesser degree of precision in language and notation (as is apparent from some of the other items on this list).
- We state five explicit axioms, employing more informal language than Barry’s, and we do not explicitly state axioms corresponding to Axioms A2 and A3 – instead we make statements without fanfare in the text.
- We accept a much looser understanding of what constitutes an **angle**, making no reference to angle-supports. We do not define the term angle. We mention reflex angles from the beginning (but make no use of them until we come to angles in circles), and quietly assume (when the time comes) that axioms that are presented by Barry in the context of wedge-angles apply also in the naturally corresponding way to reflex angles.
- When naming an angle, it is always assumed that the non-reflex angle is being referred to, unless the word “reflex” precedes or follows.

- We make no reference to results such as Pasch’s property and the “crossbar theorem”. (That is, we do not expect students to consider the necessity to prove such results or to have them given as axioms.)
- We refer to “the number of degrees” in an angle, whereas Barry treats this more correctly as “the degree-measure” of an angle.
- We take it that the definitions of parallelism, perpendicularity and “sidedness” are readily extended from lines to half-lines and line segments. (Hence, for example, we may refer to the opposite sides of a particular quadrilateral as being parallel, meaning that the lines of which they are subsets are parallel).
- We do not refer explicitly to triangles being **congruent** “under the correspondence  $(A, B, C) \rightarrow (D, E, F)$ ”, taking it instead that the correspondence is the one implied by the order in which the vertices are listed. That is, when we say “ $\triangle ABC$  is congruent to  $\triangle DEF$ ” we mean, using Barry’s terminology, “Triangle [A,B,C] is congruent to triangle [D,E,F] under the correspondence  $(A, B, C) \rightarrow (D, E, F)$ ”.
- We do not always retain the distinction in language between an angle and its measure, relying frequently instead on the context to make the meaning clear. However, we continue the practice of distinguishing notationally between the angle  $\angle ABC$  and the number  $|\angle ABC|$  of degrees in the angle<sup>1</sup>. In the same spirit, we may refer to two angles being equal, or one being equal to the sum of two others, (when we should more precisely say that the two are equal in measure, or that the measure of one is equal to the sum of the measures of the other two). Similarly, with length, we may loosely say, for example: “opposite sides of a parallelogram are equal”, or refer to “a circle of radius  $r$ ”. Where ambiguity does not arise, we may refer to angles using a single letter. That is, for example, if a diagram includes only two rays or segments from the point  $A$ , then the angle concerned may be referred to as  $\angle A$ .

Having pointed out these differences, it is perhaps worth mentioning some significant structural aspects of Barry’s geometry that are retained in our less formal version:

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<sup>1</sup>In practice, the examiners do not penalise students who leave out the bars.



- The primitive terms are almost the same, subject to the fact that their properties are conceived less formally. We treat **angle** as an extra undefined term.
- We assume that results are established in the same order as in Barry [1], up to minor local rearrangement. The exception to this is that we state all the axioms as soon as they are useful, and we bring the theorem on the angle-sum in a triangle forward to the earliest possible point (short of making it an axiom). This simplifies the proofs of a few theorems, at the expense of making it easy to see which results are theorems of so-called Neutral Geometry<sup>2</sup>.
- **Area** is not taken to be a primitive term or a given property of regions. Rather, it is defined for triangles following the establishment of the requisite result that the products of the lengths of the sides of a triangle with their corresponding altitudes are equal, and then extended to convex quadrilaterals.
- **Isometries or other transformations** are not taken as primitive. Indeed, in our case, the treatment does not extend as far as defining them. Thus they can play no role in our proofs.

## 4 Outline of the Level 2 Account

We present the account by outlining:

1. A list ( Section 5), of the terminology for the geometrical concepts. Each term in a theory is either undefined or defined, or at least definable. There have to be some undefined terms. (In textbooks, the undefined terms will be introduced by descriptions, and some of the defined terms will be given explicit definitions, in language appropriate to the level. We assume that previous level 3 work will have laid a foundation that will allow students to understand the undefined terms. We do not give the explicit definitions of all the definable terms. Instead we rely on the student's ordinary language, supplemented sometimes by informal remarks. For instance, we do not write out in cold blood the definition of the **side opposite** a given angle in a triangle, or the

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<sup>2</sup> Geometry without the axiom of parallels. This is not a concern in secondary school.

definition (in terms of set membership) of what it means to say that a line **passes through** a given point. The reason why some terms **must** be given explicit definitions is that there are alternatives, and the definition specifies the starting point; the alternative descriptions of the term are then obtained as theorems.

2. A logical account (Section 6) of the synthetic geometry theory. All the material through to LC higher is presented. The individual syllabuses will identify the relevant content by referencing it by number (e.g. Theorems 1,2, 9).
3. The geometrical constructions (Section 7) that will be studied. Again, the individual syllabuses will refer to the items on this list by number when specifying what is to be studied.
4. Some guidance on teaching (Section 8).
5. Syllabus entries for each of JC-OL, JC-HL, LC-FL, LC-OL, LC-HL.

## 5 Terms

**Undefined Terms:** angle, degree, length, line, plane, point, ray, real number, set.

**Most important Defined Terms:** area, parallel lines, parallelogram, right angle, triangle, congruent triangles, similar triangles, tangent to a circle, area.

**Other Defined terms:** acute angle, alternate angles, angle bisector, arc, area of a disc, base and corresponding apex and height of triangle or parallelogram, chord, circle, circumcentre, circumcircle, circumference of a circle, circumradius, collinear points, concurrent lines, convex quadrilateral, corresponding angles, diameter, disc, distance, equilateral triangle, exterior angles of a triangle, full angle, hypotenuse, incentre, incircle, inradius, interior opposite angles, isosceles triangle, median lines, midpoint of a segment, null angle, obtuse angle, perpendicular bisector of a segment, perpendicular lines, point of contact of a tangent, polygon, quadrilateral, radius, ratio, rectangle, reflex angle ordinary angle, rhombus, right-angled triangle, scalene triangle,

sector, segment, square, straight angle, subset, supplementary angles, transversal line, vertically-opposite angles.

**Definable terms used without explicit definition:** angles, adjacent sides, arms or sides of an angle, centre of a circle, endpoints of segment, equal angles, equal segments, line passes through point, opposite sides or angles of a quadrilateral, or vertices of triangles or quadrilaterals, point lies on line, side of a line, side of a polygon, the side opposite an angle of a triangle, vertex, vertices (of angle, triangle, polygon).

## 6 The Theory

**Line**<sup>3</sup> is short for straight line. Take a fixed **plane**<sup>4</sup>, once and for all, and consider just lines that lie in it. The plane and the lines are **sets**<sup>5</sup> of **points**<sup>6</sup>. Each line is a **subset** of the plane, i.e. each element of a line is a point of the plane. Each line is endless, extending forever in both directions. Each line has infinitely-many points. The points on a line can be taken to be ordered along the line in a natural way. As a consequence, given any three distinct points on a line, exactly one of them lies **between** the other two. Points that are not on a given line can be said to be on one or other **side** of the line. The sides of a line are sometimes referred to as **half-planes**.

**Notation 1.** We denote points by roman capital letters  $A, B, C$ , etc., and lines by lower-case roman letters  $l, m, n$ , etc.

Axioms are statements we will accept as true<sup>7</sup>.

**Axiom 1** (Two Points Axiom). *There is exactly one line through any two given points. (We denote the line through  $A$  and  $B$  by  $AB$ .)*

**Definition 1.** The line **segment**  $[AB]$  is the part of the line  $AB$  between  $A$  and  $B$  (including the endpoints). The point  $A$  divides the line  $AB$  into two pieces, called **rays**. The point  $A$  lies between all points of one ray and all

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<sup>3</sup>Line is undefined.

<sup>4</sup>Undefined term

<sup>5</sup>Undefined term

<sup>6</sup>Undefined term

<sup>7</sup> An **axiom** is a statement accepted without proof, as a basis for argument. A **theorem** is a statement deduced from the axioms by logical argument.

points of the other. We denote the ray that starts at  $A$  and passes through  $B$  by  $[AB$ . Rays are sometimes referred to as **half-lines**.

Three points usually determine three different lines.

**Definition 2.** If three or more points lie on a single line, we say they are **collinear**.

**Definition 3.** Let  $A$ ,  $B$  and  $C$  be points that are not collinear. The **triangle**  $\triangle ABC$  is the piece of the plane enclosed by the three line segments  $[AB]$ ,  $[BC]$  and  $[CA]$ . The segments are called its **sides**, and the points are called its **vertices** (singular **vertex**).

## 6.1 Length and Distance

We denote the set of all **real numbers**<sup>8</sup> by  $\mathbb{R}$ .

**Definition 4.** We denote the **distance**<sup>9</sup> between the points  $A$  and  $B$  by  $|AB|$ . We define the **length** of the segment  $[AB]$  to be  $|AB|$ .

We often denote the lengths of the three sides of a triangle by  $a$ ,  $b$ , and  $c$ . The usual thing for a triangle  $\triangle ABC$  is to take  $a = |BC|$ , i.e. the length of the side opposite the vertex  $A$ , and similarly  $b = |CA|$  and  $c = |AB|$ .

**Axiom 2** (Ruler Axiom<sup>10</sup>). *The distance between points has the following properties:*

1. *the distance  $|AB|$  is never negative;*
2.  $|AB| = |BA|$ ;
3. *if  $C$  lies on  $AB$ , between  $A$  and  $B$ , then  $|AB| = |AC| + |CB|$ ;*
4. *(marking off a distance) given any ray from  $A$ , and given any real number  $k \geq 0$ , there is a unique point  $B$  on the ray whose distance from  $A$  is  $k$ .*

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<sup>8</sup>Undefined term

<sup>9</sup>Undefined term

<sup>10</sup> Teachers used to traditional treatments that follow Euclid closely should note that this axiom (and the later Protractor Axiom) guarantees the existence of various points (and lines) without appeal to postulates about constructions using straight-edge and compass. They are powerful axioms.

**Definition 5.** The **midpoint** of the segment  $[AB]$  is the point  $M$  of the segment with <sup>11</sup>

$$|AM| = |MB| = \frac{|AB|}{2}.$$

## 6.2 Angles

**Definition 6.** A subset of the plane is **convex** if it contains the whole segment that connects any two of its points.

For example, one side of any line is a convex set, and triangles are convex sets.

We do not define the term angle formally. Instead we say: There are things called **angles**. To each angle is associated:

1. a unique point  $A$ , called its **vertex**;
2. two rays  $[AB$  and  $[AC$ , both starting at the vertex, and called the **arms** of the angle;
3. a piece of the plane called the **inside** of the angle.

An angle is either a null angle, an ordinary angle, a straight angle, a reflex angle or a full angle, Unless otherwise specified, you may take it that any angle we talk about is an ordinary angle.

**Definition 7.** An angle is a **null angle** if its arms coincide with one another and its inside is the empty set.

**Definition 8.** An angle is an **ordinary angle** if its arms are not on one line, and its inside is a convex set.

**Definition 9.** An angle is a **straight angle** if its arms are the two halves of one line, and its inside is one of the sides of that line.

**Definition 10.** An angle is a **reflex angle** if its arms are not on one line, and its inside is not a convex set.

**Definition 11.** An angle is a **full angle** if its arms coincide with one another and its inside is the rest of the plane.

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<sup>11</sup> Students may notice that the first equality implies the second.

**Definition 12.** Suppose that  $A$ ,  $B$ , and  $C$  are three noncollinear points. We denote the (ordinary) angle with arms  $[AB$  and  $[AC$  by  $\angle BAC$  (and also by  $\angle CAB$ ). We shall also use the notation  $\angle BAC$  to refer to straight angles, where  $A$ ,  $B$ ,  $C$  are collinear, and  $A$  lies between  $B$  and  $C$  (either side could be the inside of this angle).

Sometimes we want to refer to an angle without naming points, and in that case we use lower-case greek letters,  $\alpha, \beta, \gamma$ , etc.

### 6.3 Degrees

**Notation 2.** We denote the number of **degrees** in an angle  $\angle BAC$  or  $\alpha$  by the symbol  $|\angle BAC|$ , or  $|\angle \alpha|$ , as the case may be.

**Axiom 3** (Protractor Axiom). *The number of degrees in an angle (also known as its degree-measure) is always a number between  $0^\circ$  and  $360^\circ$ . The number of degrees of an ordinary angle is less than  $180^\circ$ . It has these properties:*

1. *A straight angle has  $180^\circ$ .*
2. *Given a ray  $[AB$ , and a number  $d$  between 0 and 180, there is exactly one ray from  $A$  on each side of the line  $AB$  that makes an (ordinary) angle having  $d$  degrees with the ray  $[AB$ .*
3. *If  $D$  is a point inside an angle  $\angle BAC$ , then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Null angles are assigned  $0^\circ$ , full angles  $360^\circ$ , and reflex angles have more than  $180^\circ$ . To be more exact, if  $A$ ,  $B$ , and  $C$  are noncollinear points, then the reflex angle “outside” the angle  $\angle BAC$  measures  $360^\circ - |\angle BAC|$ , in degrees.

**Definition 13.** The ray  $[AD$  is the **bisector** of the angle  $\angle BAC$  if

$$|\angle BAD| = |\angle DAC| = \frac{|\angle BAC|}{2}.$$

We say that an angle is ‘an angle of’ (for instance)  $45^\circ$ , if it has 45 degrees in it.

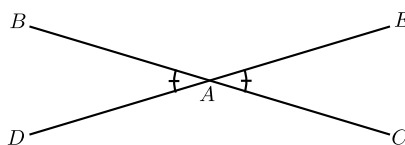
**Definition 14.** A **right angle** is an angle of exactly  $90^\circ$ .

**Definition 15.** An angle is **acute** if it has less than  $90^\circ$ , and **obtuse** if it has more than  $90^\circ$ .

**Definition 16.** If  $\angle BAC$  is a straight angle, and  $D$  is off the line  $BC$ , then  $\angle BAD$  and  $\angle DAC$  are called **supplementary angles**. They add to  $180^\circ$ .

**Definition 17.** When two lines  $AB$  and  $AC$  cross at a point  $A$ , they are **perpendicular** if  $\angle BAC$  is a right angle.

**Definition 18.** Let  $A$  lie between  $B$  and  $C$  on the line  $BC$ , and also between  $D$  and  $E$  on the line  $DE$ . Then  $\angle BAD$  and  $\angle CAE$  are called **vertically-opposite angles**.



**Figure 1.**

**Theorem 1** (Vertically-opposite Angles).

*Vertically opposite angles are equal in measure.*

*Proof.* See Figure 1. The idea is to add the same supplementary angles to both, getting  $180^\circ$ . In detail,

$$\begin{aligned} |\angle BAD| + |\angle BAE| &= 180^\circ, \\ |\angle CAE| + |\angle BAE| &= 180^\circ, \end{aligned}$$

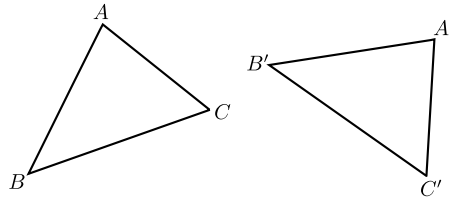
so subtracting gives:

$$\begin{aligned} |\angle BAD| - |\angle CAE| &= 0^\circ, \\ |\angle BAD| &= |\angle CAE|. \end{aligned}$$

□

## 6.4 Congruent Triangles

**Definition 19.** Let  $A, B, C$  and  $A', B', C'$  be triples of non-collinear points. We say that the triangles  $\triangle ABC$  and  $\triangle A'B'C'$  are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e.  $|AB| = |A'B'|$ ,  $|BC| = |B'C'|$ ,  $|CA| = |C'A'|$ ,  $|\angle ABC| = |\angle A'B'C'|$ ,  $|\angle BCA| = |\angle B'C'A'|$ , and  $|\angle CAB| = |\angle C'A'B'|$ . See Figure 2.



**Figure 2.**

**Notation 3.** Usually, we abbreviate the names of the angles in a triangle, by labelling them by the names of the vertices. For instance, we write  $\angle A$  for  $\angle CAB$ .

**Axiom 4** (SAS+ASA+SSS<sup>12</sup>).

If (1)  $|AB| = |A'B'|$ ,  $|AC| = |A'C'|$  and  $|\angle A| = |\angle A'|$ ,

or

(2)  $|BC| = |B'C'|$ ,  $|\angle B| = |\angle B'|$ , and  $|\angle C| = |\angle C'|$ ,

or

(3)  $|AB| = |A'B'|$ ,  $|BC| = |B'C'|$ , and  $|CA| = |C'A'|$

then the triangles  $\triangle ABC$  and  $\triangle A'B'C'$  are congruent.

**Definition 20.** A triangle is called **right-angled** if one of its angles is a right angle. The other two angles then add to  $90^\circ$ , by Theorem 4, so are both acute angles. The side opposite the right angle is called the **hypotenuse**.

**Definition 21.** A triangle is called **isosceles** if two sides are equal<sup>13</sup>. It is **equilateral** if all three sides are equal. It is **scalene** if no two sides are equal.

**Theorem 2** (Isosceles Triangles).

(1) In an isosceles triangle the angles opposite the equal sides are equal.

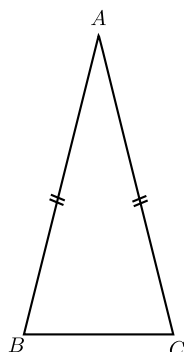
(2) Conversely, If two angles are equal, then the triangle is isosceles.

*Proof.* (1) Suppose the triangle  $\triangle ABC$  has  $AB = AC$  (as in Figure 3). Then  $\triangle ABC$  is congruent to  $\triangle ACB$  [SAS]  
 $\therefore \angle B = \angle C$ .

<sup>12</sup>It would be possible to prove all the theorems using a weaker axiom (just SAS). We use this stronger version to shorten the course.

<sup>13</sup> The simple “equal” is preferred to “of equal length”





**Figure 3.**

(2) Suppose now that  $\angle B = \angle C$ . Then  $\triangle ABC$  is congruent to  $\triangle ACB$  [ASA]  
 $\therefore |AB| = |AC|$ ,  $\triangle ABC$  is isosceles.  $\square$

*Acceptable Alternative Proof of (1).* Let  $D$  be the midpoint of  $[BC]$ , and use SAS to show that the triangles  $\triangle ABD$  and  $\triangle ACD$  are congruent. (This proof is more complicated, but has the advantage that it yields the extra information that the angles  $\angle ADB$  and  $\angle ADC$  are equal, and hence both are right angles (since they add to a straight angle)).  $\square$

## 6.5 Parallels

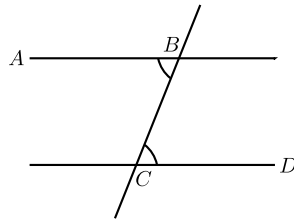
**Definition 22.** Two lines  $l$  and  $m$  are **parallel** if they are either identical, or have no common point.

**Notation 4.** We write  $l \parallel m$  for “ $l$  is parallel to  $m$ ”.

**Axiom 5** (Axiom of Parallels). *Given any line  $l$  and a point  $P$ , there is exactly one line through  $P$  that is parallel to  $l$ .*

**Definition 23.** If  $l$  and  $m$  are lines, then a line  $n$  is called a **transversal** of  $l$  and  $m$  if it meets them both.

**Definition 24.** Given two lines  $AB$  and  $CD$  and a transversal  $BC$  of them, as in Figure 4, the angles  $\angle ABC$  and  $\angle BCD$  are called **alternate** angles.

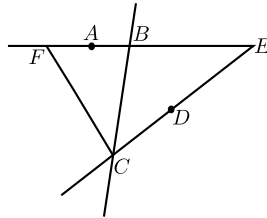


**Figure 4.**

**Theorem 3** (Alternate Angles). *Suppose that  $A$  and  $D$  are on opposite sides of the line  $BC$ .*

(1) *If  $|\angle ABC| = |\angle BCD|$ , then  $AB \parallel CD$ . In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.*

(2) *Conversely, if  $AB \parallel CD$ , then  $|\angle ABC| = |\angle BCD|$ . In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.*



**Figure 5.**

*Proof.* (1) Suppose  $|\angle ABC| = |\angle BCD|$ . If the lines  $AB$  and  $CD$  do not meet, then they are parallel, by definition, and we are done. Otherwise, they meet at some point, say  $E$ . Let us assume that  $E$  is on the same side of  $BC$  as  $D$ .<sup>14</sup> Take  $F$  on  $EB$ , on the same side of  $BC$  as  $A$ , with  $|BF| = |CE|$  (see Figure 5). [Ruler Axiom]

<sup>14</sup>Fuller detail: There are three cases:

1°:  $E$  lies on  $BC$ . Then (using Axiom 1) we must have  $E = B = C$ , and  $AB = CD$ .

2°:  $E$  lies on the same side of  $BC$  as  $D$ . In that case, take  $F$  on  $EB$ , on the same side of  $BC$  as  $A$ , with  $|BF| = |CE|$ . [Ruler Axiom]

Then  $\triangle BCE$  is congruent to  $\triangle CBF$ . [SAS]

Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

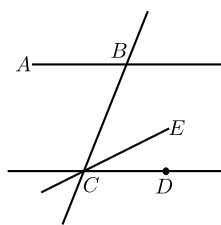
Then  $\triangle BCE$  is congruent to  $\triangle CBF$ . [SAS]  
 Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

so that  $F$  lies on  $DC$ . [Ruler Axiom]

Thus  $AB$  and  $CD$  both pass through  $E$  and  $F$ , and hence coincide, [Axiom 1]

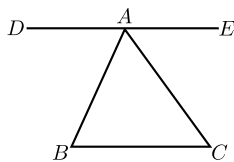
Hence  $AB$  and  $CD$  are parallel. [Definition of parallel]



**Figure 6.**

(2) To prove the converse, suppose  $AB \parallel CD$ . Pick a point  $E$  on the same side of  $BC$  as  $D$  with  $|\angle BCE| = |\angle ABC|$ . (See Figure 6.) By Part (1), the line  $CE$  is parallel to  $AB$ . By Axiom 5, there is only one line through  $C$  parallel to  $AB$ , so  $CE = CD$ . Thus  $|\angle BCD| = |\angle BCE| = |\angle ABC|$ .  $\square$

**Theorem 4** (Angle Sum 180). *The angles in any triangle add to  $180^\circ$ .*



**Figure 7.**

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so that  $F$  lies on  $DC$ . [Protractor Axiom]  
 Thus  $AB$  and  $CD$  both pass through  $E$  and  $F$ , and hence coincide. [Axiom 1]  
 3°:  $E$  lies on the same side of  $BC$  as  $A$ . Similar to the previous case.  
 Thus, in all three cases,  $AB = CD$ , so the lines are parallel.

*Proof.* Let  $\triangle ABC$  be given. Take a segment  $[DE]$  passing through  $A$ , parallel to  $BC$ , with  $D$  on the opposite side of  $AB$  from  $C$ , and  $E$  on the opposite side of  $AC$  from  $B$  (as in Figure 7). [Axiom of Parallels]

Then  $AB$  is a transversal of  $DE$  and  $BC$ , so by the Alternate Angles Theorem,

$$|\angle ABC| = |\angle DAB|.$$

Similarly,  $AC$  is a transversal of  $DE$  and  $BC$ , so

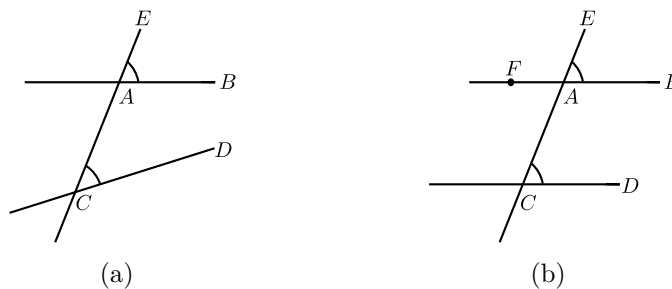
$$|\angle ACB| = |\angle CAE|.$$

Thus, using the Protractor Axiom to add the angles,

$$\begin{aligned} & |\angle ABC| + |\angle ACB| + |\angle BAC| \\ &= |\angle DAB| + |\angle CAE| + |\angle BAC| \\ &= |\angle DAE| = 180^\circ, \end{aligned}$$

since  $\angle DAE$  is a straight angle. □

**Definition 25.** Given two lines  $AB$  and  $CD$ , and a transversal  $AE$  of them, as in Figure 8(a), the angles  $\angle EAB$  and  $\angle ACD$  are called **corresponding angles**<sup>15</sup>.



**Figure 8.**

**Theorem 5 (Corresponding Angles).** *Two lines are parallel if and only if for any transversal, corresponding angles are equal.*

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<sup>15</sup>with respect to the two lines and the given transversal.

*Proof.* See Figure 8(b). We first assume that the corresponding angles  $\angle EAB$  and  $\angle ACD$  are equal. Let  $F$  be a point on  $AB$  such that  $F$  and  $B$  are on opposite sides of  $AE$ . Then we have

$$|\angle EAB| = |\angle FAC| \quad \text{[Vertically opposite angles]}$$

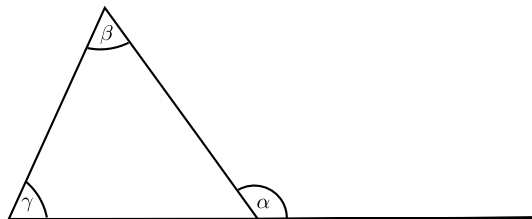
Hence the alternate angles  $\angle FAC$  and  $\angle ACD$  are equal and therefore the lines  $FA = AB$  and  $CD$  are parallel.

For the converse, let us assume that the lines  $AB$  and  $CD$  are parallel. Then the alternate angles  $\angle FAC$  and  $\angle ACD$  are equal. Since

$$|\angle EAB| = |\angle FAC| \quad \text{[Vertically opposite angles]}$$

we have that the corresponding angles  $\angle EAB$  and  $\angle ACD$  are equal.  $\square$

**Definition 26.** In Figure 9, the angle  $\alpha$  is called an **exterior angle** of the triangle, and the angles  $\beta$  and  $\gamma$  are called (corresponding) **interior opposite angles**.<sup>16</sup>



**Figure 9.**

**Theorem 6 (Exterior Angle).** *Each exterior angle of a triangle is equal to the sum of the interior opposite angles.*

*Proof.* See Figure 10. In the triangle  $\triangle ABC$  let  $\alpha$  be an exterior angle at  $A$ .

Then

$$|\alpha| + |\angle A| = 180^\circ \quad \text{[Supplementary angles]}$$

and

$$|\angle B| + |\angle C| + |\angle A| = 180^\circ. \quad \text{[Angle sum } 180^\circ\text{]}$$

Subtracting the two equations yields  $|\alpha| = |\angle B| + |\angle C|$ .  $\square$

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<sup>16</sup>The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.

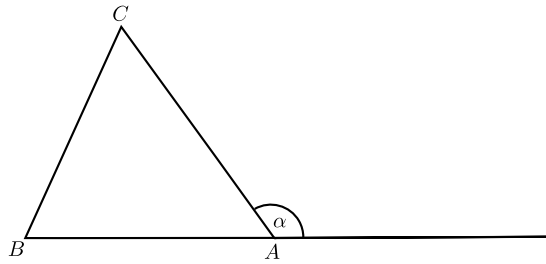


Figure 10.

**Theorem 7.**

(1) In  $\triangle ABC$ , suppose that  $|AC| > |AB|$ . Then  $|\angle ABC| > |\angle ACB|$ . In other words, the angle opposite the greater of two sides is greater than the angle opposite the lesser side.

(2) Conversely, if  $|\angle ABC| > |\angle ACB|$ , then  $|AC| > |AB|$ . In other words, the side opposite the greater of two angles is greater than the side opposite the lesser angle.

*Proof.*

(1) Suppose that  $|AC| > |AB|$ . Then take the point  $D$  on the segment  $[AC]$  with  $|AD| = |AB|$ . [Ruler Axiom]

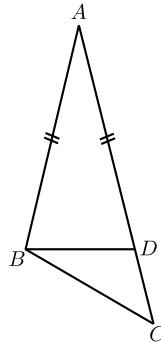


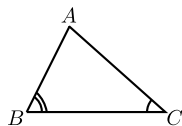
Figure 11.

See Figure 11. Then  $\triangle ABD$  is isosceles, so

$$\begin{aligned}
 |\angle ACB| &< |\angle ADB| && \text{[Exterior Angle]} \\
 &= |\angle ABD| && \text{[Isosceles Triangle]} \\
 &< |\angle ABC|.
 \end{aligned}$$

Thus  $|\angle ACB| < |\angle ABC|$ , as required.

(2)(This is a Proof by Contradiction!)  
 Suppose that  $|\angle ABC| > |\angle ACB|$ . See Figure 12.

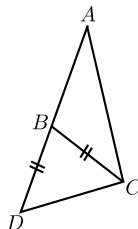


**Figure 12.**

If it could happen that  $|AC| \leq |AB|$ , then  
**either** Case 1°:  $|AC| = |AB|$ , in which case  $\triangle ABC$  is isosceles, and then  $|\angle ABC| = |\angle ACB|$ , which contradicts our assumption,  
**or** Case 2°:  $|AC| < |AB|$ , in which case Part (1) tells us that  $|\angle ABC| < |\angle ACB|$ , which also contradicts our assumption. Thus it cannot happen, and we conclude that  $|AC| > |AB|$ .  $\square$

**Theorem 8** (Triangle Inequality).

*Two sides of a triangle are together greater than the third.*



**Figure 13.**

*Proof.* Let  $\triangle ABC$  be an arbitrary triangle. We choose the point  $D$  on  $AB$  such that  $B$  lies in  $[AD]$  and  $|BD| = |BC|$  (as in Figure 13). In particular

$$|AD| = |AB| + |BD| = |AB| + |BC|.$$

Since  $B$  lies in the angle  $\angle ACD$ <sup>17</sup> we have

$$|\angle BCD| < |\angle ACD|.$$

---

<sup>17</sup> $B$  lies in a segment whose endpoints are on the arms of  $\angle ACD$ . Since this angle is  $< 180^\circ$  its inside is convex.

Because of  $|BD| = |BC|$  and the Theorem about Isosceles Triangles we have  $|\angle BCD| = |\angle BDC|$ , hence  $|\angle ADC| = |\angle BDC| < |\angle ACD|$ . By the previous theorem applied to  $\triangle ADC$  we have

$$|AC| < |AD| = |AB| + |BC|.$$

□

## 6.6 Perpendicular Lines

**Proposition 1.**<sup>18</sup> *Two lines perpendicular to the same line are parallel to one another.*

*Proof.* This is a special case of the Alternate Angles Theorem. □

**Proposition 2.** *There is a unique line perpendicular to a given line and passing through a given point. This applies to a point on or off the line.*

**Definition 27.** The **perpendicular bisector** of a segment  $[AB]$  is the line through the midpoint of  $[AB]$ , perpendicular to  $AB$ .

## 6.7 Quadrilaterals and Parallelograms

**Definition 28.** A closed chain of line segments laid end-to-end, not crossing anywhere, and not making a straight angle at any endpoint encloses a piece of the plane called a **polygon**. The segments are called the **sides** or edges of the polygon, and the endpoints where they meet are called its **vertices**. Sides that meet are called **adjacent sides**, and the ends of a side are called **adjacent vertices**. The angles at adjacent vertices are called **adjacent angles**.

**Definition 29.** A **quadrilateral** is a polygon with four vertices.

Two sides of a quadrilateral that are not adjacent are called **opposite sides**. Similarly, two angles of a quadrilateral that are not adjacent are called **opposite angles**.

---

<sup>18</sup>In this document, a proposition is a useful or interesting statement that could be proved at this point, but whose proof is not stipulated as an essential part of the programme. Teachers are free to deal with them as they see fit. For instance, they might be just mentioned, or discussed without formal proof, or used to give practice in reasoning for HLC students. It is desirable that they be mentioned, at least.



**Definition 30.** A **rectangle** is a quadrilateral having right angles at all four vertices.

**Definition 31.** A **rhombus** is a quadrilateral having all four sides equal.

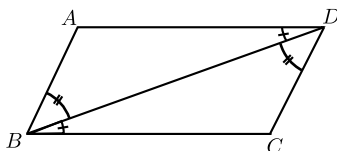
**Definition 32.** A **square** is a rectangular rhombus.

**Definition 33.** A polygon is **equilateral** if all its sides are equal, and **regular** if all its sides and angles are equal.

**Definition 34.** A **parallelogram** is a quadrilateral for which both pairs of opposite sides are parallel.

**Proposition 3.** *Each rectangle is a parallelogram.*

**Theorem 9.** *In a parallelogram, opposite sides are equal, and opposite angles are equal.*



**Figure 14.**

*Proof.* See Figure 14. Idea: Use Alternate Angle Theorem, then ASA to show that a diagonal divides the parallelogram into two congruent triangles. This gives opposite sides and (one pair of) opposite angles equal.

In more detail, let  $ABCD$  be a given parallelogram,  $AB \parallel CD$  and  $AD \parallel BC$ . Then

$$\begin{aligned} |\angle ABD| &= |\angle BDC| && \text{[Alternate Angle Theorem]} \\ |\angle ADB| &= |\angle DBC| && \text{[Alternate Angle Theorem]} \\ \Delta DAB &\text{ is congruent to } \Delta BCD. && \text{[ASA]} \end{aligned}$$

$$\therefore |AB| = |CD|, |AD| = |CB|, \text{ and } |\angle DAB| = |\angle BCD|.$$

□

**Remark 1.** Sometimes it happens that the converse of a true statement is false. For example, it is true that if a quadrilateral is a rhombus, then its diagonals are perpendicular. But it is not true that a quadrilateral whose diagonals are perpendicular is always a rhombus.

It may also happen that a statement admits several valid converses. Theorem 9 has two:

**Converse 1 to Theorem 9:** *If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram.*

*Proof.* First, one deduces from Theorem 4 that the angle sum in the quadrilateral is  $360^\circ$ . It follows that adjacent angles add to  $180^\circ$ . Theorem 3 then yields the result.  $\square$

**Converse 2 to Theorem 9:** *If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram.*

*Proof.* Drawing a diagonal, and using SSS, one sees that opposite angles are equal.  $\square$

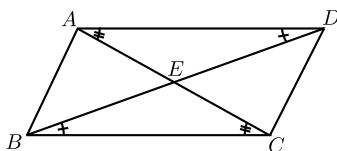
**Corollary 1.** *A diagonal divides a parallelogram into two congruent triangles.*

**Remark 2.** The converse is false: It may happen that a diagonal divides a convex quadrilateral into two congruent triangles, even though the quadrilateral is not a parallelogram.

**Proposition 4.** *A quadrilateral in which one pair of opposite sides is equal and parallel, is a parallelogram.*

**Proposition 5.** *Each rhombus is a parallelogram.*

**Theorem 10.** *The diagonals of a parallelogram bisect one another.*



**Figure 15.**

*Proof.* See Figure 15. Idea: Use Alternate Angles and ASA to establish congruence of  $\triangle ADE$  and  $\triangle CBE$ .

In detail: Let  $AC$  cut  $BD$  in  $E$ . Then

$$\begin{aligned} |\angle EAD| &= |\angle ECB| \text{ and} \\ |\angle EDA| &= |\angle EBC| && \text{[Alternate Angle Theorem]} \\ |AD| &= |BC|. && \text{[Theorem 9]} \end{aligned}$$

$\therefore \triangle ADE$  is congruent to  $\triangle CBE$ . [ASA]  $\square$

**Proposition 6** (Converse). *If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.*

*Proof.* Use SAS and Vertically Opposite Angles to establish congruence of  $\triangle ABE$  and  $\triangle CDE$ . Then use Alternate Angles.  $\square$

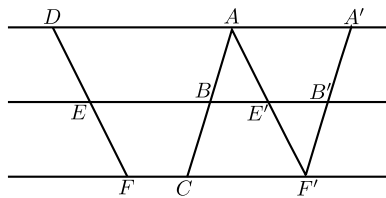
## 6.8 Ratios and Similarity

**Definition 35.** If the three angles of one triangle are equal, respectively, to those of another, then the two triangles are said to be **similar**.

**Remark 3.** Obviously, two right-angled triangles are similar if they have a common angle other than the right angle.

(The angles sum to  $180^\circ$ , so the third angles must agree as well.)

**Theorem 11.** *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*



**Figure 16.**

*Proof.* Uses opposite sides of a parallelogram, AAS, Axiom of Parallels.

In more detail, suppose  $AD \parallel BE \parallel CF$  and  $|AB| = |BC|$ . We wish to show that  $|DE| = |EF|$ .

Draw  $AE' \parallel DE$ , cutting  $EB$  at  $E'$  and  $CF$  at  $F'$ .

Draw  $F'B' \parallel AB$ , cutting  $EB$  at  $B'$ . See Figure 16.

Then

$$\begin{array}{rcll}
 |B'F'| & = & |BC| & \text{[Theorem 9]} \\
 & = & |AB|. & \text{[by Assumption]} \\
 |\angle BAE'| & = & |\angle E'F'B'|. & \text{[Alternate Angle Theorem]} \\
 |\angle AE'B| & = & |\angle F'E'B'|. & \text{[Vertically Opposite Angles]} \\
 \therefore \triangle ABE' & \text{is congruent to} & \triangle F'B'E'. & \text{[ASA]} \\
 \therefore |AE'| & = & |F'E'|. & 
 \end{array}$$

But

$$|AE'| = |DE| \text{ and } |F'E'| = |FE|. \quad \text{[Theorem 9]}$$

$$\therefore |DE| = |EF|. \quad \square$$

**Definition 36.** Let  $s$  and  $t$  be positive real numbers. We say that a point  $C$  **divides the segment**  $[AB]$  **in the ratio**  $s : t$  if  $C$  lies on the line  $AB$ , and is between  $A$  and  $B$ , and

$$\frac{|AC|}{|CB|} = \frac{s}{t}.$$

We say that a line  $l$  **cuts**  $[AB]$  **in the ratio**  $s : t$  if it meets  $AB$  at a point  $C$  that divides  $[AB]$  in the ratio  $s : t$ .

**Remark 4.** It follows from the Ruler Axiom that given two points  $A$  and  $B$ , and a ratio  $s : t$ , there is exactly one point that divides the segment  $[AB]$  in that exact ratio.

**Theorem 12.** *Let  $\triangle ABC$  be a triangle. If a line  $l$  is parallel to  $BC$  and cuts  $[AB]$  in the ratio  $s : t$ , then it also cuts  $[AC]$  in the same ratio.*

*Proof.* We prove only the commensurable case.

Let  $l$  cut  $[AB]$  in  $D$  in the ratio  $m : n$  with natural numbers  $m, n$ . Thus there are points (Figure 17)

$$D_0 = B, D_1, D_2, \dots, D_{m-1}, D_m = D, D_{m+1}, \dots, D_{m+n-1}, D_{m+n} = A,$$

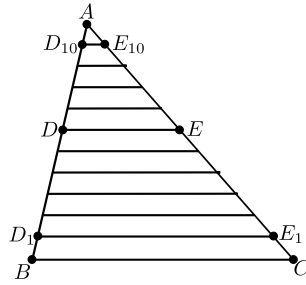


Figure 17.

equally spaced along  $[BA]$ , i.e. the segments

$$[D_0, D_1], [D_1, D_2], \dots [D_i, D_{i+1}], \dots [D_{m+n-1}, D_{m+n}]$$

have equal length.

Draw lines  $D_1E_1, D_2E_2, \dots$  parallel to  $BC$  with  $E_1, E_2, \dots$  on  $[AC]$ .

Then all the segments

$$[CE_1], [E_1E_2], [E_2E_3], \dots, [E_{m+n-1}A]$$

have the same length,

[Theorem 11]

and  $E_m = E$  is the point where  $l$  cuts  $[AC]$ .

[Axiom of Parallels]

Hence  $E$  divides  $[CA]$  in the ratio  $m : n$ .  $\square$

**Proposition 7.** *If two triangles  $\triangle ABC$  and  $\triangle A'B'C'$  have*

$$|\angle A| = |\angle A'|, \text{ and } \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|},$$

*then they are similar.*

*Proof.* Suppose  $|A'B'| \leq |AB|$ . If equal, use SAS. Otherwise, note that then  $|A'B'| < |AB|$  and  $|A'C'| < |AC|$ . Pick  $B''$  on  $[AB]$  and  $C''$  on  $[AC]$  with  $|A'B'| = |AB''|$  and  $|A'C'| = |AC''|$ . [Ruler Axiom] Then by SAS,  $\triangle A'B'C'$  is congruent to  $\triangle AB''C''$ .

Draw  $[B''D]$  parallel to  $BC$  [Axiom of Parallels], and let it cut  $AC$  at  $D$ . Now the last theorem and the hypothesis tell us that  $D$  and  $C''$  divide  $[AC]$  in the same ratio, and hence  $D = C''$ .

Thus

$$\begin{aligned} |\angle B| &= |\angle AB''C''| \text{ [Corresponding Angles]} \\ &= |\angle B'|, \end{aligned}$$

and

$$|\angle C| = |\angle AC''B''| = |\angle C'|,$$

so  $\Delta ABC$  is similar to  $\Delta A'B'C'$ .

[Definition of similar]

□

**Remark 5.** The **Converse to Theorem 12** is true:

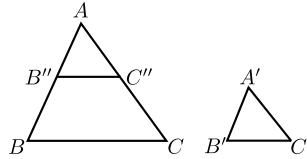
Let  $\Delta ABC$  be a triangle. If a line  $l$  cuts the sides  $AB$  and  $AC$  in the same ratio, then it is parallel to  $BC$ .

*Proof.* This is immediate from Proposition 7 and Theorem 5.

□

**Theorem 13.** If two triangles  $\Delta ABC$  and  $\Delta A'B'C'$  are similar, then their sides are proportional, in order:

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$



**Figure 18.**

*Proof.* We may suppose  $|A'B'| \leq |AB|$ . Pick  $B''$  on  $[AB]$  with  $|AB''| = |A'B'|$ , and  $C''$  on  $[AC]$  with  $|AC''| = |A'C'|$ . Refer to Figure 18. Then

$$\begin{array}{llll} \Delta AB''C'' & \text{is congruent to} & \Delta A'B'C' & \text{[SAS]} \\ \therefore |\angle AB''C''| & = & |\angle ABC| & \\ \therefore B''C'' & \parallel & BC & \text{[Corresponding Angles]} \\ \therefore \frac{|A'B'|}{|A'C'|} & = & \frac{|AB''|}{|AC''|} & \text{[Choice of } B'', C''] \\ & = & \frac{|AB|}{|AC|} & \text{[Theorem 12]} \\ \frac{|AC|}{|A'C'|} & = & \frac{|AB|}{|A'B'|} & \text{[Re-arrange]} \end{array}$$

Similarly,  $\frac{|BC|}{|B'C'|} = \frac{|AB|}{|A'B'|}$

□

**Proposition 8** (Converse). *If*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|},$$

*then the two triangles  $\triangle ABC$  agus  $\triangle A'B'C'$  are similar.*

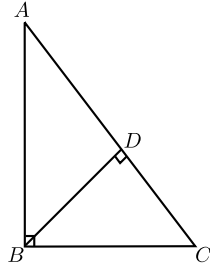
*Proof.* Refer to Figure 18. If  $|A'B'| = |AB|$ , then by SSS the two triangles are congruent, and therefore similar. Otherwise, assuming  $|A'B'| < |AB|$ , choose  $B''$  on  $AB$  and  $C''$  on  $AC$  with  $|AB''| = |A'B'|$  and  $|AC''| = |A'C'|$ . Then by Proposition 7,  $\triangle AB''C''$  is similar to  $\triangle ABC$ , so

$$|B''C''| = |AB''| \cdot \frac{|BC|}{|AB|} = |A'B'| \cdot \frac{|BC|}{|AB|} = |B'C'|.$$

Thus by SSS,  $\triangle A'B'C'$  is congruent to  $\triangle A''B''C''$ , and hence similar to  $\triangle ABC$ .  $\square$

## 6.9 Pythagoras

**Theorem 14** (Pythagoras). *In a right-angle triangle the square of the hypotenuse is the sum of the squares of the other two sides.*



**Figure 19.**

*Proof.* Let  $\triangle ABC$  have a right angle at  $B$ . Draw the perpendicular  $BD$  from the vertex  $B$  to the hypotenuse  $AC$  (shown in Figure 19).

The right-angle triangles  $\triangle ABC$  and  $\triangle ADB$  have a common angle at  $A$ .  $\therefore \triangle ABC$  is similar to  $\triangle ADB$ .

$$\therefore \frac{|AC|}{|AB|} = \frac{|AB|}{|AD|},$$

so

$$|AB|^2 = |AC| \cdot |AD|.$$

Similarly,  $\triangle ABC$  is similar to  $\triangle BDC$ .

$$\therefore \frac{|AC|}{|BC|} = \frac{|BC|}{|DC|},$$

so

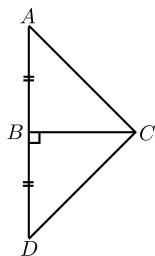
$$|BC|^2 = |AC| \cdot |DC|.$$

Thus

$$\begin{aligned} |AB|^2 + |BC|^2 &= |AC| \cdot |AD| + |AC| \cdot |DC| \\ &= |AC| (|AD| + |DC|) \\ &= |AC| \cdot |AC| \\ &= |AC|^2. \end{aligned}$$

□

**Theorem 15** (Converse to Pythagoras). *If the square of one side of a triangle is the sum of the squares of the other two, then the angle opposite the first side is a right angle.*



**Figure 20.**

*Proof.* (Idea: Construct a second triangle on the other side of  $[BC]$ , and use Pythagoras and SSS to show it congruent to the original.)

In detail: We wish to show that  $|\angle ABC| = 90^\circ$ .

Draw  $BD \perp BC$  and make  $|BD| = |AB|$  (as shown in Figure 20).



Then

$$\begin{aligned}
 |DC| &= \sqrt{|DC|^2} \\
 &= \sqrt{|BD|^2 + |BC|^2} && \text{[Pythagoras]} \\
 &= \sqrt{|AB|^2 + |BC|^2} && [ |AB| = |BD| ] \\
 &= \sqrt{|AC|^2} && \text{[Hypothesis]} \\
 &= |AC|.
 \end{aligned}$$

$\therefore \triangle ABC$  is congruent to  $\triangle DBC$ . [SSS]  
 $\therefore |\angle ABC| = |\angle DBC| = 90^\circ$ . □

**Proposition 9 (RHS).** *If two right angled triangles have hypotenuse and another side equal in length, respectively, then they are congruent.*

*Proof.* Suppose  $\triangle ABC$  and  $\triangle A'B'C'$  are right-angle triangles, with the right angles at  $B$  and  $B'$ , and have hypotenuses of the same length,  $|AC| = |A'C'|$ , and also have  $|AB| = |A'B'|$ . Then by using Pythagoras' Theorem, we obtain  $|BC| = |B'C'|$ , so by SSS, the triangles are congruent. □

**Proposition 10.** *Each point on the perpendicular bisector of a segment  $[AB]$  is equidistant from the ends.*

**Proposition 11.** *The perpendiculars from a point on an angle bisector to the arms of the angle have equal length.*

## 6.10 Area

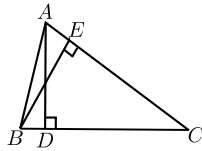
**Definition 37.** If one side of a triangle is chosen as the base, then the opposite vertex is the **apex** corresponding to that base. The corresponding **height** is the length of the perpendicular from the apex to the base. This perpendicular segment is called an **altitude** of the triangle.

**Theorem 16.** *For a triangle, base times height does not depend on the choice of base.*

*Proof.* Let  $AD$  and  $BE$  be altitudes (shown in Figure 21). Then  $\triangle BCE$  and  $\triangle ACD$  are right-angled triangles that share the angle  $C$ , hence they are similar. Thus

$$\frac{|AD|}{|BE|} = \frac{|AC|}{|BC|}.$$

Re-arrange to yield the result. □



**Figure 21.**

**Definition 38.** The **area** of a triangle is half the base by the height.

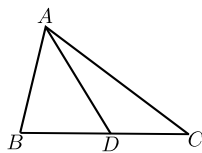
**Notation 5.** We denote the area by “area of  $\triangle ABC$ ”<sup>19</sup>.

**Proposition 12.** *Congruent triangles have equal areas.*

**Remark 6.** This is another example of a proposition whose converse is false. It may happen that two triangles have equal area, but are not congruent.

**Proposition 13.** *If a triangle  $\triangle ABC$  is cut into two by a line  $AD$  from  $A$  to a point  $D$  on the segment  $[BC]$ , then the areas add up properly:*

$$\text{area of } \triangle ABC = \text{area of } \triangle ABD + \text{area of } \triangle ADC.$$



**Figure 22.**

*Proof.* See Figure 22. All three triangles have the same height, say  $h$ , so it comes down to

$$\frac{|BC| \times h}{2} = \frac{|BD| \times h}{2} + \frac{|DC| \times h}{2},$$

which is obvious, since

$$|BC| = |BD| + |DC|.$$

□

---

<sup>19</sup>  $|\triangle ABC|$  will also be accepted.

If a figure can be cut up into nonoverlapping triangles (i.e. triangles that either don't meet, or meet only along an edge), then its area is taken to be the sum of the area of the triangles<sup>20</sup>.

If figures of equal areas are added to (or subtracted from) figures of equal areas, then the resulting figures also have equal areas<sup>21</sup>.

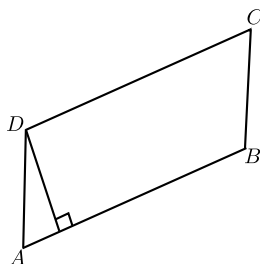
**Proposition 14.** *The area of a rectangle having sides of length  $a$  and  $b$  is  $ab$ .*

*Proof.* Cut it into two triangles by a diagonal. Each has area  $\frac{1}{2}ab$ . □

**Theorem 17.** *A diagonal of a parallelogram bisects the area.*

*Proof.* A diagonal cuts the parallelogram into two congruent triangles, by Corollary 1. □

**Definition 39.** Let the side  $AB$  of a parallelogram  $ABCD$  be chosen as a base (Figure 23). Then the **height** of the parallelogram **corresponding to that base** is the height of the triangle  $\triangle ABC$ .



**Figure 23.**

**Proposition 15.** *This height is the same as the height of the triangle  $\triangle ABD$ , and as the length of the perpendicular segment from  $D$  onto  $AB$ .*

---

<sup>20</sup> If students ask, this does not lead to any ambiguity. In the case of a convex quadrilateral,  $ABCD$ , one can show that

$$\text{area of } \triangle ABC + \text{area of } \triangle CDA = \text{area of } \triangle ABD + \text{area of } \triangle BCD.$$

In the general case, one proves the result by showing that there is a common refinement of any two given triangulations.

<sup>21</sup> Follows from the previous footnote.

**Theorem 18.** *The area of a parallelogram is the base by the height.*

*Proof.* Let the parallelogram be  $ABCD$ . The diagonal  $BD$  divides it into two triangles,  $\triangle ABD$  and  $\triangle CDB$ . These have equal area, [Theorem 17] and the first triangle shares a base and the corresponding height with the parallelogram. So the areas of the two triangles add to  $2 \times \frac{1}{2} \times \text{base} \times \text{height}$ , which gives the result.  $\square$

## 6.11 Circles

**Definition 40.** A **circle** is the set of points at a given distance (its **radius**) from a fixed point (its **centre**). Each line segment joining the centre to a point of the circle is also called a **radius**. The plural of radius is radii. A **chord** is the segment joining two points of the circle. A **diameter** is a chord through the centre. All diameters have length twice the radius. This number is also called **the diameter** of the circle.

Two points  $A, B$  on a circle cut it into two pieces, called **arcs**. You can specify an arc uniquely by giving its endpoints  $A$  and  $B$ , and one other point  $C$  that lies on it. A **sector** of a circle is the piece of the plane enclosed by an arc and the two radii to its endpoints.

The length of the whole circle is called its **circumference**. For every circle, the circumference divided by the diameter is the same. This ratio is called  $\pi$ .

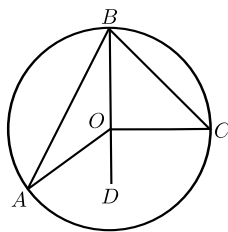
A **semicircle** is an arc of a circle whose ends are the ends of a diameter.

Each circle divides the plane into two pieces, the inside and the outside. The piece inside is called a **disc**.

If  $B$  and  $C$  are the ends of an arc of a circle, and  $A$  is another point, not on the arc, then we say that the angle  $\angle BAC$  is the angle at  $A$  **standing on the arc**. We also say that it **stands on the chord**  $[BC]$ .

**Theorem 19.** *The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.*

*Proof.* There are several cases for the diagram. It will be sufficient for students to examine one of these. The idea, in all cases, is to draw the line through the centre and the point on the circumference, and use the Isosceles Triangle Theorem, and then the Protractor Axiom (to add or subtract angles, as the case may be).



**Figure 24.**

In detail, for the given figure, Figure 24, we wish to show that  $|\angle AOC| = 2|\angle ABC|$ .

Join  $B$  to  $O$  and continue the line to  $D$ . Then

$$\begin{aligned}
 |OA| &= |OB|. && \text{[Definition of circle]} \\
 \therefore |\angle BAO| &= |\angle ABO|. && \text{[Isosceles triangle]} \\
 \therefore |\angle AOD| &= |\angle BAO| + |\angle ABO| && \text{[Exterior Angle]} \\
 &= 2 \cdot |\angle ABO|.
 \end{aligned}$$

Similarly,

$$|\angle COD| = 2 \cdot |\angle CBO|.$$

Thus

$$\begin{aligned}
 |\angle AOC| &= |\angle AOD| + |\angle COD| \\
 &= 2 \cdot |\angle ABO| + 2 \cdot |\angle CBO| \\
 &= 2 \cdot |\angle ABC|.
 \end{aligned}$$

□

**Corollary 2.** *All angles at points of the circle, standing on the same arc, are equal. In symbols, if  $A, A', B$  and  $C$  lie on a circle, and both  $A$  and  $A'$  are on the same side of the line  $BC$ , then  $\angle BAC = \angle BA'C$ .*

*Proof.* Each is half the angle subtended at the centre. □

**Remark 7.** The converse is true, but one has to be careful about sides of  $BC$ :

**Converse to Corollary 2:** *If points  $A$  and  $A'$  lie on the same side of the line  $BC$ , and if  $|\angle BAC| = |\angle BA'C|$ , then the four points  $A, A', B$  and  $C$  lie on a circle.*

*Proof.* Consider the circle  $s$  through  $A, B$  and  $C$ . If  $A'$  lies outside the circle, then take  $A''$  to be the point where the segment  $[A'B]$  meets  $s$ . We then have

$$|\angle BA'C| = |\angle BAC| = |\angle BA''C|,$$

by Corollary 2. This contradicts Theorem 6.

A similar contradiction arises if  $A'$  lies inside the circle. So it lies on the circle.  $\square$

**Corollary 3.** *Each angle in a semicircle is a right angle. In symbols, if  $BC$  is a diameter of a circle, and  $A$  is any other point of the circle, then  $\angle BAC = 90^\circ$ .*

*Proof.* The angle at the centre is a straight angle, measuring  $180^\circ$ , and half of that is  $90^\circ$ .  $\square$

**Corollary 4.** *If the angle standing on a chord  $[BC]$  at some point of the circle is a right angle, then  $[BC]$  is a diameter.*

*Proof.* The angle at the centre is  $180^\circ$ , so is straight, and so the line  $BC$  passes through the centre.  $\square$

**Definition 41.** A **cyclic** quadrilateral is one whose vertices lie on some circle.

**Corollary 5.** *If  $ABCD$  is a cyclic quadrilateral, then opposite angles sum to  $180^\circ$ .*

*Proof.* The two angles at the centre standing on the same arcs add to  $360^\circ$ , so the two halves add to  $180^\circ$ .  $\square$

**Remark 8.** The converse also holds: *If  $ABCD$  is a convex quadrilateral, and opposite angles sum to  $180^\circ$ , then it is cyclic.*

*Proof.* This follows directly from Corollary 5 and the converse to Corollary 2.  $\square$

It is possible to approximate a disc by larger and smaller equilateral polygons, whose area is as close as you like to  $\pi r^2$ , where  $r$  is its radius. For this reason, we say that the area of the disc is  $\pi r^2$ .

**Proposition 16.** *If  $l$  is a line and  $s$  a circle, then  $l$  meets  $s$  in zero, one, or two points.*

*Proof.* We classify by comparing the length  $p$  of the perpendicular from the centre to the line, and the radius  $r$  of the circle. If  $p > r$ , there are no points. If  $p = r$ , there is exactly one, and if  $p < r$  there are two.  $\square$

**Definition 42.** The line  $l$  is called a **tangent** to the circle  $s$  when  $l \cap s$  has exactly one point. The point is called the **point of contact** of the tangent.

**Theorem 20.**

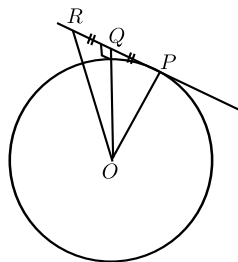
(1) *Each tangent is perpendicular to the radius that goes to the point of contact.*

(2) *If  $P$  lies on the circle  $s$ , and a line  $l$  is perpendicular to the radius to  $P$ , then  $l$  is tangent to  $s$ .*

*Proof.* (1) This proof is a proof by contradiction.

Suppose the point of contact is  $P$  and the tangent  $l$  is not perpendicular to  $OP$ .

Let the perpendicular to the tangent from the centre  $O$  meet it at  $Q$ . Pick  $R$  on  $PQ$ , on the other side of  $Q$  from  $P$ , with  $|QR| = |PQ|$  (as in Figure 25).



**Figure 25.**

Then  $\triangle OQR$  is congruent to  $\triangle OQP$ . [SAS]

$$\therefore |OR| = |OP|,$$

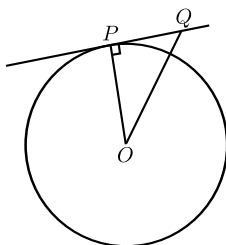
so  $R$  is a second point where  $l$  meets the circle. This contradicts the given fact that  $l$  is a tangent.

Thus  $l$  must be perpendicular to  $OP$ , as required.

(2) (Idea: Use Pythagoras. This shows directly that each other point on  $l$  is further from  $O$  than  $P$ , and hence is not on the circle.)

In detail: Let  $Q$  be any point on  $l$ , other than  $P$ . See Figure 26. Then

$$\begin{aligned} |OQ|^2 &= |OP|^2 + |PQ|^2 && \text{[Pythagoras]} \\ &> |OP|^2. \\ \therefore |OQ| &> |OP|. \end{aligned}$$



**Figure 26.**

$\therefore Q$  is not on the circle. [Definition of circle]

$\therefore P$  is the only point of  $l$  on the circle.

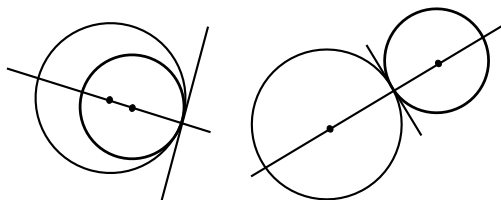
$\therefore l$  is a tangent. [Definition of tangent]

□

**Corollary 6.** *If two circles share a common tangent line at one point, then the two centres and that point are collinear.*

*Proof.* By part (1) of the theorem, both centres lie on the line passing through the point and perpendicular to the common tangent. □

The circles described in Corollary 6 are shown in Figure 27.



**Figure 27.**

**Remark 9.** Any two distinct circles will intersect in 0, 1, or 2 points.

If they have two points in common, then the common chord joining those two points is perpendicular to the line joining the centres.

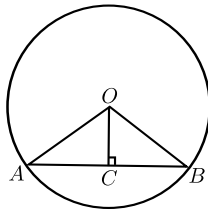
If they have just one point of intersection, then they are said to be *touching* and this point is referred to as their *point of contact*. The centres and the point of contact are collinear, and the circles have a common tangent at that point.



**Theorem 21.**

- (1) *The perpendicular from the centre to a chord bisects the chord.*
- (2) *The perpendicular bisector of a chord passes through the centre.*

*Proof.* (1) (Idea: Two right-angled triangles with two pairs of sides equal.)  
See Figure 28.



**Figure 28.**

In detail:

$$\begin{aligned} |OA| &= |OB| && \text{[Definition of circle]} \\ |OC| &= |OC| \end{aligned}$$

$$\begin{aligned} |AC| &= \sqrt{|OA|^2 - |OC|^2} && \text{[Pythagoras]} \\ &= \sqrt{|OB|^2 - |OC|^2} \\ &= |CB|. && \text{[Pythagoras]} \end{aligned}$$

$\therefore \triangle OAC$  is congruent to  $\triangle OBC$ . [SSS]  
 $\therefore |AC| = |CB|$ .

(2) This uses the Ruler Axiom, which has the consequence that a segment has exactly one midpoint.

Let  $C$  be the foot of the perpendicular from  $O$  on  $AB$ .

By Part (1),  $|AC| = |CB|$ , so  $C$  is the midpoint of  $[AB]$ .

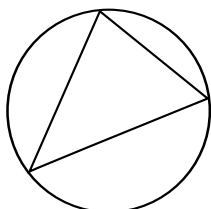
Thus  $CO$  is the perpendicular bisector of  $AB$ .

Hence the perpendicular bisector of  $AB$  passes through  $O$ . □

## 6.12 Special Triangle Points

**Proposition 17.** *If a circle passes through three non-collinear points  $A$ ,  $B$ , and  $C$ , then its centre lies on the perpendicular bisector of each side of the triangle  $\triangle ABC$ .*

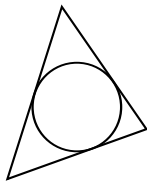
**Definition 43.** The **circumcircle** of a triangle  $\triangle ABC$  is the circle that passes through its vertices (see Figure 29). Its centre is the **circumcentre** of the triangle, and its radius is the **circumradius**.



**Figure 29.**

**Proposition 18.** *If a circle lies inside the triangle  $\triangle ABC$  and is tangent to each of its sides, then its centre lies on the bisector of each of the angles  $\angle A$ ,  $\angle B$ , and  $\angle C$ .*

**Definition 44.** The **incircle** of a triangle is the circle that lies inside the triangle and is tangent to each side (see Figure 30). Its centre is the **incentre**, and its radius is the **inradius**.



**Figure 30.**

**Proposition 19.** *The lines joining the vertices of a triangle to the centre of the opposite sides meet in one point.*

**Definition 45.** A line joining a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. The point where the three medians meet is called the **centroid**.

**Proposition 20.** *The perpendiculars from the vertices of a triangle to the opposite sides meet in one point.*

**Definition 46.** The point where the perpendiculars from the vertices to the opposite sides meet is called the **orthocentre** (see Figure 31).

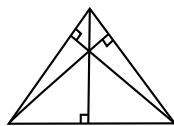


Figure 31.

## 7 Constructions to Study

The instruments that may be used are:

**straight-edge:** This may be used (together with a pencil) to draw a straight line passing through two marked points.

**compass:** This instrument allows you to draw a circle with a given centre, passing through a given point. It also allows you to take a given segment  $[AB]$ , and draw a circle centred at a given point  $C$  having radius  $|AB|$ .

**ruler:** This is a straight-edge marked with numbers. It allows you measure the length of segments, and to mark a point  $B$  on a given ray with vertex  $A$ , such that the length  $|AB|$  is a given positive number. It can also be employed by sliding it along a set square, or by other methods of sliding, while keeping one or two points on one or two curves.

**protractor:** This allows you to measure angles, and mark points  $C$  such that the angle  $\angle BAC$  made with a given ray  $[AB$  has a given number of degrees. It can also be employed by sliding it along a line until some line on the protractor lies over a given point.

**set-squares:** You may use these to draw right angles, and angles of  $30^\circ$ ,  $60^\circ$ , and  $45^\circ$ . It can also be used by sliding it along a ruler until some coincidence occurs.

The prescribed constructions are:

1. Bisector of a given angle, using only compass and straight edge.
2. Perpendicular bisector of a segment, using only compass and straight edge.
3. Line perpendicular to a given line  $l$ , passing through a given point not on  $l$ .

4. Line perpendicular to a given line  $l$ , passing through a given point on  $l$ .
5. Line parallel to given line, through given point.
6. Division of a segment into 2, 3 equal segments, without measuring it.
7. Division of a segment into any number of equal segments, without measuring it.
8. Line segment of given length on a given ray.
9. Angle of given number of degrees with a given ray as one arm.
10. Triangle, given lengths of three sides.
11. Triangle, given SAS data.
12. Triangle, given ASA data.
13. Right-angled triangle, given the length of the hypotenuse and one other side.
14. Right-angled triangle, given one side and one of the acute angles (several cases).
15. Rectangle, given side lengths.
16. Circumcentre and circumcircle of a given triangle, using only straight-edge and compass.
17. Incentre and incircle of a given triangle, using only straight-edge and compass.
18. Angle of  $60^\circ$ , without using a protractor or set square.
19. Tangent to a given circle at a given point on it.
20. Parallelogram, given the length of the sides and the measure of the angles.
21. Centroid of a triangle.
22. Orthocentre of a triangle.

## 8 Teaching Approaches

### 8.1 Practical Work

Practical exercises and experiments should be undertaken before the study of theory. These should include:

1. Lessons along the lines suggested in the Guidelines for Teachers [2]. We refer especially to Section 4.6 (7 lessons on Applied Arithmetic and Measure), Section 4.9 (14 lessons on Geometry), and Section 4.10 (4 lessons on Trigonometry).
2. Lessons along the lines of Prof. Barry's memo.
3. Ideas from Technical Drawing.
4. Material in [3].

### 8.2 From Discovery to Proof

It is intended that all of the geometrical results on the course would first be encountered by students through investigation and discovery. As a result of various activities undertaken, students should come to appreciate that certain features of certain shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features therefore seem to be general results that we have reason to believe might always be true. At this stage in the work, we ask students to accept them as true for the purpose of applying them to various contextualised and abstract problems, but we also agree to come back later to revisit this question of their truth. Nonetheless, even at this stage, students should be asked to consider whether investigating a number of examples in this way is sufficient to be convinced that a particular result always holds, or whether a more convincing argument is required. Is a person who refuses to believe that the asserted result will always be true being unreasonable? An investigation of a statement that appears at first to be always true, but in fact is not, may be helpful, (e.g. the assertion that  $n^2 + n + 41$  is prime for all  $n \in \mathbb{N}$ ). Reference might be made to other examples of conjectures that were historically believed to be true until counterexamples were found.

Informally, the ideas involved in a mathematical proof can be developed even at this investigative stage. When students engage in activities that lead

to closely related results, they may readily come to appreciate the manner in which these results are connected to each other. That is, they may see for themselves or be led to see that the result they discovered today is an inevitable logical consequence of the one they discovered yesterday. Also, it should be noted that working on problems or cuts involves logical deduction from general results.

Later, students at the relevant levels need to proceed beyond accepting a result on the basis of examples towards the idea of a more convincing logical argument. Informal justifications, such as a dissection-based proof of Pythagoras theorem, have a role to play here. Such justifications develop an argument more strongly than a set of examples. It is worth discussing what the word prove means in various contexts, such as in a criminal trial, or in a civil court, or in everyday language. What mathematicians regard as a proof is quite different from these other contexts. The logic involved in the various steps must be unassailable. One might present one or more of the readily available dissection-based proofs of fallacies and then probe a dissection-based proof of Pythagoras theorem to see what possible gaps might need to be bridged.

As these concepts of argument and proof are developed, students should be led to appreciate the need to formalise our idea of a mathematical proof to lay out the ground rules that we can all agree on. Since a formal proof only allows us to progress logically from existing results to new ones, the need for axioms is readily identified, and the students can be introduced to formal proofs.

## 9 Syllabus for JCOL

### 9.1 Concepts

Set, plane, point, line, ray, angle, real number, length, degree, triangle, right-angle, congruent triangles, similar triangles, parallel lines, parallelogram, area, tangent to a circle, subset, segment, collinear points, distance, midpoint of a segment, reflex angle, ordinary angle, straight angle, null angle, full angle, supplementary angles, vertically-opposite angles, acute angle, obtuse angle, angle bisector, perpendicular lines, perpendicular bisector of a segment, ratio, isosceles triangle, equilateral triangle, scalene triangle, right-angled triangle, exterior angles of a triangle, interior opposite angles, hypotenuse, alternate

angles, corresponding angles, polygon, quadrilateral, convex quadrilateral, rectangle, square, rhombus, base and corresponding apex and height of triangle or parallelogram, transversal line, circle, radius, diameter, chord, arc, sector, circumference of a circle, disc, area of a disc, circumcircle, point of contact of a tangent, vertex, vertices (of angle, triangle, polygon), endpoints of segment, arms of an angle, equal segments, equal angles, adjacent sides, angles, or vertices of triangles or quadrilaterals, the side opposite an angle of a triangle, opposite sides or angles of a quadrilateral, centre of a circle.

## 9.2 Constructions

Students will study constructions 1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15.

## 9.3 Axioms and Proofs

The students should be exposed to some formal proofs. They will not be examined on these. They will see Axioms 1,2,3,4,5, and study the proofs of Theorems 1, 2, 3, 4, 5, 6, 9, 10, 13 (statement only), 14, 15; and direct proofs of Corollaries 3, 4.

# 10 Syllabus for JCHL

## 10.1 Concepts

Those for JCOL, and concurrent lines.

## 10.2 Constructions

Students will study all the constructions prescribed for JC-OL, and also constructions 3 and 7.

## 10.3 Logic, Axioms and Theorems

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

They will study Axioms 1, 2, 3, 4, 5. They will study the proofs of Theorems 1, 2, 3, 4\*, 5, 6\*, 9\*, 10, 11, 12, 13, 14\*, 15, 19\*, Corollaries 1,

2, 3, 4, 5, and their converses. Those marked with a \* may be asked in examination.

The formal material on area will not be studied at this level. Students will deal with area only as part of the material on arithmetic and mensuration.

## 11 Syllabus for LCFL

A knowledge of the theorems prescribed for JC-OL will be assumed, and questions based on them may be asked in examination. Proofs will not be required.

### 11.1 Constructions

A knowledge of the constructions prescribed for JC-OL will be assumed, and may be examined. In addition, students will study constructions 18, 19, 20.

## 12 Syllabus for LCOL

### 12.1 Constructions

A knowledge of the constructions prescribed for JC-OL will be assumed, and may be examined. In addition, students will study the constructions prescribed for LC-FL, and constructions 16, 17, 21.

### 12.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-OL will be assumed. (In the transitional period, students who have taken the discontinued JL-FL, will have to study these as well.)

Students will study proofs of Theorems 7, 8, 11, 12, 13, 16, 17, 18, 20, 21, and Corollary 6.

No proofs are examinable. Students will be examined using problems that can be attacked using the theory.



## 13 Syllabus for LCHL

### 13.1 Constructions

A knowledge of the constructions prescribed for JC-HL will be assumed, and may be examined. In addition, students will study the constructions prescribed for LC-OL, and construction 22.

### 13.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies, is equivalent to, if and only if, proof by contradiction.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-HL will be assumed.

Students will study all the theorems and corollaries prescribed for LC-OL, but will not, in general, be asked to reproduce their proofs in examination.

However, they may be asked to give proofs of the Theorems 11, 12, 13, concerning ratios, which lay the proper foundation for the proof of Pythagoras studied at JC.

They will be asked to solve geometrical problems (so-called “cuts”) and write reasoned accounts of the solutions. These problems will be such that they can be attacked using the given theory. The study of the propositions may be a useful way to prepare for such examination questions.

## References

- [1] Patrick D. Barry. *Geometry with Trigonometry*. Horwood. Chichester. 2001. ISBN 1-898563-69-1.
- [2] Junior Cycle Course Committee, NCCA. *Mathematics: Junior Certificate Guidelines for Teachers*. Stationary Office, Dublin. 2002. ISBN 0-7557-1193-9.
- [3] Fiacre O’Cairbre, John McKeon, and Richard O. Watson. *A Resource for Transition Year Mathematics Teachers*. DES. Dublin. 2006.

- [4] Anthony G. O'Farrell. *School Geometry*. IMTA Newsletter 109 (2009) 21-28.

## **Section C**

**The following syllabus material is retained from the previous Junior Certificate Mathematics syllabus published in 2000.**

# 1. INTRODUCTION

## 1.1 Context

Mathematics is a wide-ranging subject with many aspects. On the one hand, in its manifestations in the form of counting, measurement, pattern and geometry, it permeates the natural and constructed world about us, and provides the basic language and techniques for handling many aspects of everyday and scientific life. On the other hand, it deals with abstractions, logical arguments, and fundamental ideas of truth and beauty, and so is an intellectual discipline and a source of aesthetic satisfaction. These features have caused it to be given names such as “the queen and the servant of the sciences”. Its role in education reflects this dual nature: it is both practical and theoretical—geared to applications and of intrinsic interest—with the two elements firmly interlinked.

Mathematics has traditionally formed a substantial part of the education of young people in Ireland throughout their schooldays. Its value to all students as a component of general education, and as preparation for life after school, has been recognised by the community at large. Accordingly, it is of particular importance that the mathematical education offered to students should be appropriate to their abilities, needs and interests, and should fully and appositely reflect the broad nature of the subject and its potential for enhancing the students’ development.

## 1.2 Aims

It is intended that mathematics education should:

### A. Contribute to the personal development of the students:

- helping them to acquire the mathematical knowledge, skills and understanding necessary for personal fulfilment;
- developing their problem-solving skills and creative talents, and introducing them to ideas of modelling;
- developing their ability to handle abstractions and generalisations, and to recognise and present logical arguments;
- furthering their powers of communication, both oral and written, and thus their ability to share ideas with other people;
- fostering their appreciation of the creative and aesthetic aspects of mathematics, and their recognition and enjoyment of mathematics in the world around them;
- hence, enabling them to develop a positive attitude towards mathematics as an interesting and valuable subject of study.

### B. Help to provide them with the mathematical knowledge, skills and understanding needed for continuing their education, and eventually for life and work:

- promoting their confidence and competence in using the mathematical knowledge and skills required for everyday life, work and leisure;
- equipping them for the study of other subjects in school;
- preparing a firm foundation for appropriate studies later on;
- in particular, providing a basis for further education in mathematics itself.

### 1.3 General objectives

The teaching and learning of mathematics has been described as involving *facts*, *skills*, *concepts* (or “conceptual structures”), *strategies*, and—stemming from these—*appreciation*.

In terms of student outcomes, this can be formulated as follows. The students should be able to *recall* relevant facts. They should be able to demonstrate *instrumental understanding* (“knowing how”) and necessary *psychomotor* skills (skills of physical coordination). They should possess *relational understanding* (“knowing why”). They should be able to *apply* their knowledge in familiar and eventually in unfamiliar contexts; and they should develop *analytical* and *creative* powers in mathematics. Hence, they should develop *appreciative* attitudes to the subject and its uses. The aims listed in Section 1.2 can therefore be translated into the following general objectives.

**A.** Students should be able to *recall* basic facts; that is, they should be able to:

- display knowledge of conventions such as terminology and notation;
- recognise basic geometrical figures and graphical displays;
- state important derived facts resulting from their studies.

(Thus, they should have fundamental information readily available to enhance understanding and aid application.)

**B.** They should be able to demonstrate *instrumental understanding*; hence, they should know how (and when) to:

- carry out routine computational procedures and other such algorithms;
- perform measurements and constructions to an appropriate degree of accuracy;
- present information appropriately in tabular, graphical and pictorial form, and read information presented in these forms;
- use mathematical equipment such as calculators, rulers, set squares, protractors and compasses, as required for these procedures.

(Thus, they should be equipped with the basic competencies needed for mathematical activities.)

**C.** They should have acquired *relational understanding*; that is, understanding of concepts and conceptual structures, so that they can:

- interpret mathematical statements;
- interpret information presented in tabular, graphical and pictorial form;
- recognise patterns, relationships and structures;
- follow mathematical reasoning.

(Thus, they should be able to see mathematics as an integrated, meaningful and logical discipline.)

**D.** They should be able to *apply* their knowledge of facts and skills; that is, when working in familiar types of context, they should be able to:

- translate information presented verbally into mathematical form;
- select and use appropriate mathematical formulae or techniques in order to process the information;
- draw relevant conclusions.

(Thus, they should be able to use mathematics and recognise that it has many areas of applicability.)

- E.** They should be able to *analyse* information, including information presented in cross-curricular and unfamiliar contexts; hence, they should be able to:
- select appropriate strategies leading to the solution of problems;
  - form simple mathematical models;
  - justify conclusions.
- F.** They should be able to *create* mathematics for themselves; that is, they should be able to:
- explore patterns;
  - formulate conjectures;
  - support, communicate and explain findings.
- G.** They should have developed the *psychomotor* skills necessary for all the tasks described above.
- H.** They should be able to *communicate* mathematics, both verbally and in written form; that is, they should be able to:
- describe and explain the mathematical procedures they undertake;
  - explain findings and justify conclusions (as indicated above).
- I.** They should *appreciate* mathematics as a result of being able to:
- use mathematical methods successfully;
  - recognise mathematics throughout the curriculum and in their environment;
  - apply mathematics successfully to common experience;
  - acknowledge the beauty of form, structure and pattern;
  - share mathematical experiences with other people.
- J.** They should be *aware* of the history of mathematics and hence of its past, present and future role as part of our culture.

## 2. HIGHER LEVEL

### Rationale

The Higher course is geared to the needs of students of above average mathematical ability. Among the students taking the course are those who will proceed with their study of advanced mathematics not only for the Leaving Certificate but also at third level; some are the mathematicians of the next generation. However, not all students taking the course are future specialists or even future users of academic mathematics. Moreover, when they start to study the material, some are only beginning to be able to deal with abstract concepts.

A balance must be struck, therefore, between challenging the most able students and encouraging those who are developing a little more slowly. Provision must be made not only for the academic student of the future, but also for the citizen of a society in which mathematics appears in, and is applied to, everyday life. The course therefore focuses on material that underlies academic mathematical studies, ensuring that students have a chance to develop their mathematical abilities and interests to a high level; but it also covers the more practical and obviously applicable topics that students are meeting in their lives outside school.

For the target group, particular emphasis can be placed on the development of powers of abstraction and generalisation and on an introduction to the idea of proof—hence giving students a feeling for the great mathematical concepts that span many centuries and cultures. Problem-solving can be addressed in both mathematical and applied contexts. Alongside this, adequate attention must be paid to the acquisition and consolidation of fundamental skills, in the absence of which the students' development and progress will be handicapped.

### Aims

In the light of the general aims of mathematics education listed in section 1.2, the specific aims are that the Higher course will provide students with the following:

- a firm understanding of mathematical concepts and relationships;
- confidence and competence in basic skills;
- the ability to formulate and solve problems;
- an introduction to the idea of proof and to the role of logical argument in building up a mathematical system;
- a developing appreciation of the power and beauty of mathematics and of the manner in which it provides a useful and efficient system for the formulation and solution of problems.

### Assessment objectives

The assessment objectives are objectives A, B, C, D (dealing with knowledge, understanding and application), G (dealing with psychomotor skills) and H (dealing with communication). These objectives should be interpreted in the context of the aims of the Higher course as formulated above.

### Content

Knowledge of the content of the primary curriculum is assumed, but many concepts and skills are revisited for treatment at greater depth and at a greater level of difficulty or abstraction.

*It is assumed that calculators and mathematical tables are available for appropriate use.*

## Sets

1. Listing of elements of a set. Membership of a set defined by a rule. Universe, subsets. Null set (empty set). Equality of sets.
2. Venn diagrams.
3. Set operations: intersection, union, difference, complement. Set operations extended to three sets.
4. Commutative property and associative property for intersection and union; failure of commutativity and associativity for difference; necessity of brackets for the non-associative operation of difference. Distributive property of union over intersection and of intersection over union; necessity of brackets.

Not envisaged as examination terminology.

## Number systems

1. The set **N** of natural numbers. Order ( $<, \leq, >, \geq$ ). Place value. Sets of divisors. Pairs of factors. Prime numbers. Sets of multiples. Lowest common multiple. Highest common factor. Cardinal number of a set.

The operations of addition, subtraction, multiplication and division in **N**. Meaning of  $a^n$  for  $a, n \in \mathbf{N}, n \neq 0$ . Estimation leading to approximate answers.

2. The set **Z** of integers. Order ( $<, \leq, >, \geq$ ).

The operations of addition, subtraction, multiplication and division in **Z**. Use of the number line to illustrate addition, subtraction and multiplication. Meaning of  $a^n$  for  $a \in \mathbf{Z}, n \in \mathbf{N}, n \neq 0$ . Estimation leading to approximate answers.

3. The set **Q** of rational numbers. Decimals, fractions, percentages. Decimals and fractions plotted on the number line.

Rational numbers expressed as decimals. Terminating decimals expressed as fractions.

The operations of addition, subtraction, multiplication and division in **Q**. Rounding off. Significant figures for integer values only. Estimation leading to approximate answers.

Ratio and proportion.



4. Meaning of  $a^p$  where  $a, p \in \mathbf{Q}$ .

Rules for indices (where  $a, b, p, q \in \mathbf{Q}$  and  $a, b \neq 0$ ):

$$a^p a^q = a^{p+q}$$

$$\frac{a^p}{a^q} = a^{p-q}$$

$$a^0 = 1$$

$$(a^p)^q = a^{pq}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}, \quad n \in \mathbf{Z}, n \neq 0, a > 0$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, \quad m, n \in \mathbf{Z}, n \neq 0, a > 0$$

$$a^{-p} = \frac{1}{a^p}$$

$$(ab)^p = a^p b^p$$

$$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$$

Square roots, reciprocals: understanding and computation.

Scientific notation: non-zero positive rationals expressed in the form  $a \times 10^n$ , where  $n \in \mathbf{Z}$  and  $1 \leq a < 10$ .

5. The set  $\mathbf{R}$  of real numbers: every point on the number line represents a real number. Order ( $<, \leq, >, \geq$ ).

Addition, subtraction and multiplication applied to  $a \pm \sqrt{b}$ , where  $a \in \mathbf{Q}, b \in \mathbf{Q}^+$ .

The set of irrational numbers  $\mathbf{R} \setminus \mathbf{Q}$ .

6. Commutative and associative properties for addition and multiplication; failure of commutativity and associativity for subtraction and division; distributive property of multiplication over addition.

Not envisaged as examination terminology.

Priority of operations.

### Applied arithmetic and measure

1. Bills. Profit and loss. Percentage profit. Percentage discount. Tax. Annual interest. Compound interest (interest added at regular intervals to a maximum of three; formula not required). Value added tax (VAT).

Percentage profit based on cost price or selling price (relevant one to be specified in examination questions).

2. SI units of length (m), area (m<sup>2</sup>), volume (m<sup>3</sup>), mass (kg), and time (s). Multiples and submultiples. Twenty-four hour clock, transport timetables. Relationship between average speed, distance and time.

Multiples and submultiples: mm, cm, km, cm<sup>2</sup>, hectare, km<sup>2</sup>, cm<sup>3</sup>, g, tonne, minute, hour. Use of “litre”.

3. Perimeter.

Area: square, rectangle, triangle.

Surface area and volume of rectangular solids (i.e. solids with uniform rectangular cross-section).

Derivation and use of the relevant formulae for perimeter, area and volume.

$$\frac{\text{Length of circumference of circle}}{\text{Length of diameter}} = \pi.$$

Use of formulae for length of circumference of circle ( $2\pi r$ ) and for area of disc (i.e. area of region enclosed by circle,  $\pi r^2$ ). Use of formulae for curved surface area and volume of cylinder ( $2\pi rh$ ,  $\pi r^2 h$ ), right circular cone ( $\pi rl$ ,  $\frac{1}{3}\pi r^2 h$ ) and sphere ( $4\pi r^2$ ,  $\frac{4}{3}\pi r^3$ ).

Application to problems, including use of the Theorem of Pythagoras.

Problems may include compound figures made out of those specified above.

## Algebra

1. Meaning of variable, constant, term, expression, coefficient.

Evaluation of expressions.

2. Addition and subtraction of simple algebraic expressions of forms such as:

$$(ax + by + c) \pm \dots \pm (dx + ey + f)$$

$$(ax^2 + bx + c) \pm \dots \pm (dx^2 + ex + f)$$

where  $a, b, c, d, e, f \in \mathbf{Z}$ .

Examples:

$$(2x + 3) + (4x - 2)$$

$$(3x + 2y) - (x + 3y - 4)$$

$$(5x^2 + 7x - 2) + (2x^2 - x - 7)$$

Use of the associative and distributive property to simplify such expressions as:

$$a(bx + cy + d) + \dots + e(fx + gy + h)$$

$$a(bx^2 + cx + d)$$

$$ax(bx^2 + c)$$

where  $a, b, c, d, e, f, g, h \in \mathbf{Z}$ .

Examples:

$$3(x + 4) - 5(2x + 3) +$$

$$2(5x - 6)$$

$$5(3x^2 - 4x + 8)$$

Multiplication of expressions of forms such as:

$$(ax + b)(cx + d)$$

$$(ax + b)(cx^2 + dx + e)$$

where  $a, b, c, d, e \in \mathbf{Z}$ .

Examples:

$$(2x - 3)(5x + 4)$$

$$(x - 4)(x^2 - 5x - 11)$$

Division of expressions of forms such as:

$$(ax^2 + bx + c) \div (ex + f)$$

$$(ax^3 + bx^2 + cx + d) \div (ex + f)$$

where  $a, b, c, d, e, f \in \mathbf{Z}$ .

Examples:

$$(2x^2 + 11x + 15) \div (x + 3)$$

$$(6x^2 + x - 12) \div (3x - 4)$$

$$(6x^3 - x^2 - 33x - 28) \div (3x + 4)$$

Rearrangement of formulae.

Addition and subtraction of expressions of the form:

$$\frac{ax + b}{c} \pm \dots \pm \frac{dx + e}{f}$$

where  $a, b, c, d, e, f \in \mathbf{Z}$ ,

$$\frac{a}{bx + c} \pm \frac{p}{qx + r}$$

where  $a, b, c, p, q, r \in \mathbf{Z}$ .

3. Use of the distributive property in the factorising of expressions such as:

$$abxy + ay$$

where  $a, b \in \mathbf{Z}$ ,

$$sx - ty + tx - sy$$

where  $s, t, x, y$  are variable.

Example:  $8xy - 4y$

Factorisation of quadratic expressions of the form:

$$ax^2 + bx$$

$$ax^2 + bx + c$$

where  $a, b, c \in \mathbf{Z}$ .

Difference of two squares of the form  $a^2x^2 - b^2y^2$ , where  $a, b \in \mathbf{N}$ .

4. Formation and interpretation of number sentences leading to the solution of first degree equations in one variable.

First degree equations in two variables. Problems and their solutions.

Quadratic equations of the form  $ax^2 + bx + c = 0$ . Solution using factors and/or the formula for real roots only. Problems and their solutions.

5. Equations of the form:

$$\frac{ax + b}{c} \pm \dots \pm \frac{dx + e}{f} = \frac{g}{h}$$

where  $a, b, c, d, e, f, g, h \in \mathbf{Z}$ ,

$$\frac{a}{bx + c} \pm \frac{p}{qx + r} = \frac{d}{e}$$

where  $a, b, c, p, q, r, d, e \in \mathbf{Z}$ . Problems and their solutions.

6. Solution of linear inequalities in one variable, of forms such as:

$$ax + b \leq c$$

$$a \leq bx + c < d$$

where  $a, b, c, d \in \mathbf{Z}$ .

Examples:

$$2x - 1 \leq 9$$

$$1 \leq 2x - 1 < 11$$

$$3 > 2x - 7 > -5$$

## Functions and graphs

1. Concept of a function. Couples, domain, codomain, range.

2. Use of function notation:

$$f(x) =$$

$$f: x \rightarrow$$

$$y =$$

Drawing graphs of functions  $f: x \rightarrow f(x)$ , where  $f(x)$  is of the form  $ax + b$  or  $ax^2 + bx + c$ , where  $a, b, c \in \mathbf{Z}$ ,  $x \in \mathbf{R}$ .

Using the graphs to estimate the (range of) value(s) of  $x$  for which  $f(x) = k$ , where  $k \in \mathbf{R}$ .

3. Maximum and minimum values of quadratic functions estimated from graphs.
4. Graphing solution sets on the number line for linear inequalities in one variable.
5. Graphical treatment of solution of first degree simultaneous equations in two variables.

Solution of quadratic inequalities is excluded, but students may be asked to read off a range of values for which a function is (say) negative.

### 3. ORDINARY LEVEL

#### **Rationale**

The Ordinary course is geared to the needs of students of average mathematical ability. Typically, when such students come in to second level schools, some are only beginning to be able to deal with abstract ideas and some are not yet ready to do so. However, many of them may eventually go on to use and apply mathematics—perhaps even quite advanced mathematics—in their future careers, and all of them will meet the subject to a greater or lesser degree in their daily lives.

The Ordinary course, therefore, must start where these students are, offering mathematics that is meaningful and accessible to them at their present stage of development. It should also provide for the gradual introduction of more abstract ideas, leading the students towards the use of academic mathematics in the context of further study. The course therefore pays considerable attention to consolidating the foundation laid at primary level and to addressing practical topics; but it also covers aspects of the traditional mathematical areas of algebra, geometry, trigonometry and functions.

For the target group, particular emphasis can be placed on the development of mathematics as a body of knowledge and skills that makes sense and that can be used in many different ways—hence, as an efficient system for the solution of problems and provision of answers. Alongside this, adequate attention must be paid to the acquisition and consolidation of fundamental skills, in the absence of which the students' development and progress will be handicapped.

#### **Aims**

In the light of the general aims of mathematics education listed in section 1.2, the specific aims are that the Ordinary course will provide students with the following:

- an understanding of mathematical concepts and of their relationships;
- confidence and competence in basic skills;
- the ability to solve problems;
- an introduction to the idea of logical argument;
- appreciation both of the intrinsic interest of mathematics and of its usefulness and efficiency for formulating and solving problems.

#### **Assessment objectives**

The assessment objectives are objectives A, B, C, D (dealing with knowledge, understanding and application), G (dealing with psychomotor skills) and H (dealing with communication). These objectives should be interpreted in the context of the aims of the Ordinary course as formulated above.

#### **Content**

The content of the primary curriculum is taken as a prerequisite, but many concepts and skills are revisited for treatment at greater depth and at a greater level of difficulty or, ultimately, of abstraction.

*It is assumed that calculators and mathematical tables are available for appropriate use.*

## Sets

1. Listing of elements of a set. Membership of a set defined by a rule. Universe, subsets. Null set (empty set). Equality of sets.
2. Venn diagrams.
3. Set operations: intersection, union, difference, complement. Set operations extended to three sets.
4. Commutative property and associative property for intersection and union; failure of commutativity and associativity for difference; necessity of brackets for the non-associative operation of difference.

Not envisaged as examination terminology.

## Number systems

1. The set  $\mathbf{N}$  of natural numbers. Order ( $<, \leq, >, \geq$ ). Place value. Sets of divisors. Pairs of factors. Prime numbers. Sets of multiples. Lowest common multiple. Highest common factor. Cardinal number of a set.

The operations of addition, subtraction, multiplication and division in  $\mathbf{N}$ . Meaning of  $a^n$  for  $a, n \in \mathbf{N}, n \neq 0$ . Estimation leading to approximate answers.

2. The set  $\mathbf{Z}$  of integers. Order ( $<, \leq, >, \geq$ ).

The operations of addition, subtraction, multiplication and division in  $\mathbf{Z}$ . Use of the number line to illustrate addition, subtraction and multiplication. Meaning of  $a^n$  for  $a \in \mathbf{Z}, n \in \mathbf{N}, n \neq 0$ . Estimation leading to approximate answers.

3. The set  $\mathbf{Q}$  of rational numbers. Decimals, fractions, percentages. Decimals and fractions plotted on the number line.

The operations of addition, subtraction, multiplication and division in  $\mathbf{Q}$ . Rounding off. Estimation leading to approximate answers.

Ratio and proportion.

4. Rules for indices (where  $a \in \mathbf{Q}$ ,  $m, n \in \mathbf{N}$ ,  $m \neq 0$ ,  $n \neq 0$ ):

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}, m > n$$

$$(a^m)^n = a^{mn}$$

Meaning of  $a^{\frac{1}{2}}$ , where  $a \geq 0$ .

Square roots, reciprocals: understanding and computation.

Scientific notation: non-zero positive rationals expressed in the form  $a \times 10^n$ , where  $n \in \mathbf{N}$  and  $1 \leq a < 10$ .

5. The set  $\mathbf{R}$  of real numbers: the idea that every point on the number line represents a real number. Order ( $<$ ,  $\leq$ ,  $>$ ,  $\geq$ ).

6. Commutative and associative properties for addition and multiplication; failure of commutativity and associativity for subtraction and division; distributive property of multiplication over addition.

Not envisaged as examination terminology.

Priority of operations.

### Applied arithmetic and measure

1. Bills. Profit and loss. Percentage profit. Percentage discount. Tax. Annual interest. Compound interest (interest added at regular intervals to a maximum of three; formula not required). Value added tax (VAT).

Percentage profit based on cost price or selling price (relevant one to be specified in examination questions).

2. SI units of length (m), area ( $\text{m}^2$ ), volume ( $\text{m}^3$ ), mass (kg), and time (s). Multiples and submultiples. Twenty-four hour clock, transport timetables. Relationship between average speed, distance and time.

Multiples and submultiples: mm, cm, km,  $\text{cm}^2$ , hectare,  $\text{km}^2$ ,  $\text{cm}^3$ , g, tonne, minute, hour. Use of “litre”.

3. Perimeter.

Area: square, rectangle, triangle.

Surface area and volume of rectangular solids (i.e. solids with uniform rectangular cross-section).

$$\frac{\text{Length of circumference of circle}}{\text{Length of diameter}} = \pi.$$

Use of formulae for length of circumference of circle ( $2\pi r$ ) and for area of disc (i.e. area of region enclosed by circle,  $\pi r^2$ ).  
 Use of formulae for curved surface area and volume of cylinder ( $2\pi rh$ ,  $\pi r^2 h$ ) and sphere ( $4\pi r^2$ ,  $\frac{4}{3}\pi r^3$ ).

Problems may include compound figures made out of those specified above.

Application to problems.

## Algebra

### 1. Meaning of variable, constant, term, expression, coefficient.

Evaluation of expressions

### 2. Addition and subtraction of simple algebraic expressions of forms such as:

$$(ax + by + c) \pm \dots \pm (dx + ey + f)$$

$$(ax^2 + bx + c) \pm \dots \pm (dx^2 + ex + f)$$

where  $a, b, c, d, e, f \in \mathbf{Z}$ .

Use of the associative and distributive property to simplify such expressions as:

$$a(bx + cy + d) + \dots + e(fx + gy + h)$$

$$a(bx^2 + cx + d)$$

$$ax(bx^2 + c)$$

where  $a, b, c, d, e, f, g, h \in \mathbf{Z}$ .

Multiplication of expressions of the form:

$$(ax + b)(cx + d)$$

$$(ax + b)(cx^2 + dx + e)$$

where  $a, b, c, d, e \in \mathbf{Z}$ .

Addition and subtraction of expressions of the form

$$\frac{ax + b}{c} \pm \dots \pm \frac{dx + e}{f}$$

where  $a, b, c, d, e, f \in \mathbf{Z}$ .

### 3. Use of the distributive law in the factorising of expressions such as:

$$abxy + ay$$

where  $a, b \in \mathbf{Z}$ ,

$$sx - ty + tx - sy$$

where  $s, t, x, y$  are variable.

Factorisation of quadratic expressions of the form:

$$ax^2 + bx$$

$$x^2 + bx + c$$

where  $a, b, c \in \mathbf{Z}$

Examples:

$$(2x + 3) + (4x - 2)$$

$$(3x + 2y) - (x + 3y - 4)$$

$$(5x^2 + 7x - 2) +$$

$$(2x^2 - x - 7)$$

Examples:

$$3(x + 4) - 5(2x + 3) + 2(x + 3) +$$

$$2(5x - 6)$$

$$y(2x + 1)$$

Examples:

$$(2x - 3)(5x + 4)$$

$$(x - 4)(x^2 - 5x - 11)$$

Example:  $8xy - 4y$



Difference of two squares. Simple examples

Examples:  $x^2 - y^2$   
 $x^2 - 16$   
 $9 - y^2$

4. Formation and interpretation of number sentences leading to the solution of first degree equations in one variable.

First degree equations in two variables, with coefficients elements of  $\mathbf{Z}$  and solutions also elements of  $\mathbf{Z}$ . Problems and their solutions.

Quadratic equations of the form  $x^2 + bx + c = 0$  where  $b, c \in \mathbf{Z}$  and  $x^2 + bx + c$  is factorisable. Solution of simple problems leading to quadratic equations.

5. Solution of equations of the form

$$\frac{ax + b}{c} \pm \dots \pm \frac{dx + e}{f} = \frac{g}{h}$$

where  $a, b, c, d, e, f, g, h \in \mathbf{Z}$ .

6. Solution of linear inequalities in one variable, of forms such as  $ax + b \leq c$ , where  $a, b, c \in \mathbf{Z}$ ,  $x \in \mathbf{Z}$ .

Examples:  $2x - 1 \leq 9$   
 $10 - 2x > 2$

## Functions and graphs

1. Concept of a function. Couples, domain, codomain, range.

2. Use of function notation:

$$f(x) =$$

$$f : x \rightarrow$$

$$y =$$

Drawing graphs of functions  $f : x \rightarrow f(x)$ , where  $f(x)$  is of the form  $ax + b$  or  $ax^2 + bx + c$

where  $a, b, c \in \mathbf{Z}$ ,  $x \in \mathbf{R}$ .

Using the graphs to estimate solution of equations of the type  $f(x) = 0$ .

3. Graphing solution sets on the number line for linear inequalities in one variable.

Example:  $2x + 1 < 5, x \in \mathbf{R}$

4. Graphical treatment of solution of first degree simultaneous equations in two variables.



## 4. FOUNDATION LEVEL

### Rationale

The Foundation course is geared to the needs of students who are unready for or unsuited by the mathematics of the Ordinary course. Some are not yet at a developmental stage at which they can deal with abstract concepts; some may have encountered difficulties in adjusting to post-primary school and may need a particularly gradual introduction to second level work; some have learning styles that essentially do not match the traditional approach of post-primary schools. Many of the students may still be uncomfortable with material presented in the later stages of the primary curriculum. Nonetheless, they need to learn to cope with mathematics in everyday life and perhaps in further study.

The Foundation course must therefore help the students to construct a clearer knowledge of, and to develop improved skills in, basic mathematics, and to develop an awareness of its usefulness. Appropriate new material should also be introduced, so that the students can feel that they are making a fresh start and are progressing. The course therefore pays great attention to consolidating the foundation laid at primary level and to addressing practical issues; but it also covers new topics and lays a foundation for progress to more traditional study in the areas of algebra, geometry and functions. An appeal is made to different interests and learning styles, for example by paying attention to visual and spatial as well as numerical aspects.

For the target group, particular emphasis can be placed on promoting students' confidence in themselves (confidence that they can do mathematics) and in the subject (confidence that mathematics makes sense). Thus, attention must be paid to the acquisition and consolidation of fundamental skills, as indicated above; and concepts should be embedded in meaningful contexts. Many opportunities can thus be presented for students to achieve success.

### Aims

In the light of the general aims of mathematics education listed in section 1.2, the specific aims are that the Foundation course will provide students with the following:

- an understanding of basic mathematical concepts and relationships;
- confidence and competence in basic skills;
- the ability to solve simple problems;
- experience of following clear arguments and of citing evidence to support their own ideas;
- appreciation of mathematics both as an enjoyable activity through which they experience success and as a useful body of knowledge and skills.

### Assessment objectives

The assessment objectives are objectives A, B, C, D (dealing with knowledge, understanding and application), G (dealing with psychomotor skills) and H (dealing with communication). These objectives should be interpreted in the context of the aims of the Foundation course as formulated above.

### Content

The content of the primary curriculum is taken as a prerequisite, but many concepts and skills are revisited for revision and for treatment at a greater depth or level of difficulty.

*It is assumed that calculators and mathematical tables are available for appropriate use.*

## Sets

1. Listing of elements of a set. Membership of a set defined by a rule. Universe, subsets. Null set (empty set). Equality of sets.
2. Venn diagrams.
3. Set operations: intersection and union (for two sets only), complement.
4. Commutative property for intersection and union.

Not envisaged as examination terminology.

## Number systems

1. The set  $\mathbf{N}$  of natural numbers. Order ( $<, \leq, >, \geq$ ). Idea of place value. Sets of multiples. Lowest common multiple.

The operations of addition, subtraction, multiplication and division in  $\mathbf{N}$  where the answer is in  $\mathbf{N}$ . Meaning of  $a^n$  for  $a, n \in \mathbf{N}, n \neq 0$ . Evaluation of expressions containing at most one level of brackets.

Estimation leading to approximate answers.

Examples:

$$2 + 7(4 - 1)$$

$$6 + 10 \times 3$$

$$3(14 - 5) - (7 + 2)$$

2. The set  $\mathbf{Z}$  of integers. Positional order on the number line.

The operation of addition in  $\mathbf{Z}$ .

3. The set  $\mathbf{Q}^+$  of positive rational numbers.

Fractions: emphasis on fractions having 2, 3, 4, 7, 8, 16, 5, 10, 100 and 1000 as denominators. Equivalent fractions. The operations of addition, subtraction and multiplication in  $\mathbf{Q}^+$ . Estimation leading to approximate answers. Fractions expressed as decimals; for computations without a calculator, computation for fractions with the above denominators excluding 3, 7 and 16.

Decimals: place value. The operations of addition, subtraction, multiplication and division. Rounding off to not more than three decimal places. Estimation leading to approximate answers.

Percentage: fraction to percentage. Suitable fractions and decimals expressed as percentages.

Example:  $\frac{32}{100}$ ; 32%

Equivalence of fractions, decimals and percentages.

Example:  $\frac{42}{100}$ ; 0.42; 42%

4. Squares and square roots.
5. Commutative property.  
Priority of operations.

Not envisaged as examination terminology.

### Applied arithmetic and measure

1. Bills: shopping; electricity, telephone, gas, etc. Value added tax (VAT). Applications to meter readings and to fixed and variable charges. Percentage profit: to calculate selling price when given the cost price and the percentage profit or loss; to calculate the percentage profit or loss when given the cost and selling prices. Percentage discount. Compound interest for not more than three years. Calculating income tax.
2. SI units of length (m), area (m<sup>2</sup>), volume (m<sup>3</sup>), mass (kg), and time (s). Multiples and submultiples. Twenty-four hour clock, transport timetables. Relationship between average speed, distance and time.
3. Calculating distance from a map. Use of scales on drawings.
4. Perimeter.

Percentage profit based on cost price or selling price (relevant one to be specified in examination questions).  
Also: percentage increase. e.g. 5% increase in attendance at a match.

Multiples and submultiples: mm, cm, km, g, cm<sup>2</sup>, km<sup>2</sup>, cm<sup>3</sup>, minute, hour.  
Use of "litre".  
Students should be familiar with everyday use of "weight".

Area: square, rectangle, triangle.

Volume of rectangular solids (i.e. solids with uniform rectangular cross-section).

$$\frac{\text{Length of circumference of circle}}{\text{Length of diameter}} = \pi.$$

Use of formulae for length of circumference of circle ( $2\pi r$ ), for area of disc (i.e. area of region enclosed by circle,  $\pi r^2$ ).  
Use of formula for volume of cylinder ( $\pi r^2 h$ ).

## Algebra

1. Formulae, idea of an unknown, idea of a variable.

Evaluation of expressions of forms such as  $ax + by$  and  $a(x + y)$  where  $a, b, x, y \in \mathbf{N}$ ; evaluation of quadratic expressions of the form  $x^2 + ax + b$  where  $a, b, x \in \mathbf{N}$ .

2. Use of associative and distributive properties to simplify expressions of forms such as:

$$a(x \pm b) + c(x \pm d)$$

$$x(x \pm a) + b(x \pm c)$$

where  $a, b, c, d, x \in \mathbf{N}$ .

3. Solution of first degree equations in one variable where the solution is a natural number.

Informal treatment (see *Guidelines for Teachers*.)

Examples:

Find the value of  $3x + 7y$  and of  $6(x + y)$  for given values of  $x$  and  $y$ .

Find the value of  $x^2 + 5x + 7$  when  $x = 4$ .

Examples:

$$3(x - 2) + 2(x + 1)$$

$$x(x + 1) + 2(x + 2)$$

(see *Guidelines for Teachers*).

Examples:

$$\text{Solve } 3x + 4 = 19.$$

$$\text{Solve } 4(x - 1) = 12.$$

## Relations, functions and graphs

1. Couples. Use of arrow diagrams to illustrate relations.

Example: "is greater than".

2. Plotting points. Joining points to form a line.

3. Drawing the graph of forms such as  $y = ax + b$  for a specified range of values of  $x$ , where  $a, b \in \mathbf{N}$ . Simple interpretation of the graph.

Example: Draw the graph of  $y = 3x + 5$  from  $x = 1$  to  $x = 6$ .

## 5. ASSESSMENT

**N.B. The details below apply to the retained syllabus material only.**

### Introduction

Guidelines for assessment of the course are specified as follows.

- Assessment of the course is based on the following general principles:
  - the status and standing of the Junior Certificate should be maintained;
  - candidates should be able to demonstrate what they know rather than what they do not know;
  - examinations should build candidates' confidence in their ability to do mathematics;
  - full coverage of both knowledge and skills should be encouraged.
- Written examination at the end of the Junior Cycle can test the following objectives (see section 1.3): objectives A to D, G and H, dealing respectively with recall, instrumental understanding, relational understanding and application, together with the appropriate psychomotor (physical) and communication skills.
- In interpreting the objectives suitably for students at each level, the aims of the relevant course should be borne in mind (see section 2.2, 3.2, or 4.2, as appropriate).

### Design of examinations

These guidelines lead to the following points regarding the design of examinations.

- The choice of questions offered should be such as to encourage full coverage of the course and to promote equity in the tasks undertaken by different students.
- Each question in each paper should display a suitable gradient of difficulty.

Typically, this is achieved by three-part questions with:

- an easy first part;
- a second part of moderate difficulty;
- a final part of greater difficulty.

With regard to the objectives, typically:

- the first part tests recall or very simple manipulation;
- the second part tests the choice and execution of routine procedures or constructions, or interpretation;
- the third part tests application.

Typically also, the three parts of the question should test cognate areas.

- Questions should be grouped by broad topic so that students encounter work in a familiar setting; but it is not intended that the same sub-topic should always appear in exactly the same place in the paper.
- In formulating questions:
  - the language used should be simple and direct;
  - the symbolism should be easily interpreted;
  - diagrams should be reasonably accurate, but in general no information should be communicated solely by a diagram.

### Grade criteria

Knowledge and skills displayed by the students can be related to standards of achievement, as reflected in the different grades awarded for the Junior Certificate examinations. For details pertaining to grade criteria, see *Guidelines for Teachers*.









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